Exact Set-valued Estimation using Constrained Convex Generators for uncertain Linear Systems ⋆

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Abstract: Set-valued state estimation when in the presence of uncertainties in the model have been addressed in the literature essentially following three main approaches: i) interval arithmetic of the uncertain dynamics with the estimates; ii) factorizing the uncertainty into matrices with unity rank; and, iii) performing the convex hull for the vertices of the uncertainty space. Approach i) and ii) introduce a lot of conservatism because both disregard the relationship of the parameters with the entries of the dynamics matrix. On the other hand, approach iii) has a large growth on the number of variables required to represent the set or is approximated losing its main advantage in comparison with i) and ii). In this paper, with the application of autonomous vehicles in GPS-denied areas that resort to beacon signals for localization, we develop an exact (meaning no added conservatism) and optimal (smallest growth in the number of variables) closed-form definition for the convex hull of Convex Constrained Generators (CCGs). This results in a more efficient method to represent the minimum volume convex set corresponding to the state estimation. Given that reductions methods are still lacking in the literature for CCGs, we employ an approximation using ray-shooting that is comparable in terms of accuracy with methods for Constrained Zonotopes as the ones implemented in CORA. Simulations illustrate the greater accuracy of CCGs with the proposed convex hull operation in comparison to Constrained Zonotopes.

Keywords: Observers for linear systems; Parameter-varying systems; Guidance navigation and control.

1. INTRODUCTION

Missions where autonomous vehicles have onboard sensors to help their localization or use a guaranteed state estimation filter to perform collision avoidance Ribeiro et al. (2020, 2021) can benefit from having very accurate set representations as conservative estimates would translate in very restricted movement control signals. Incorporating noise-corrupted range and bearing measurements is typically done in the literature through an over-approximation of the set resulting from range-only measurements by intervals Jaulin (2011) or using ellipsoids Marcal et al. (2005). The development of Constrained Convex Generators (CCGs) in Silvestre (2022a) allowed modeling this type of measurements for linear models with no uncertainties. CCGs have the advantage of allowing the representation of circular or ellipsoidal shapes (like the sets obtained using range measurements) as well as intersections of different convex bodies (like polytopes obtained from the propagation of a uniform initial uncertain using a linear model).

The state estimation task can also be carried following the stochastic approach with Kalman filters that vary depending on the assumptions. Single beacon range measurement was tackled in Batista et al. (2011) by a transformation of the nonlinear dynamics to obtain a Linear Time Varying (LTV) which allows for a Kalman Filter. The nonlinear model can be directly used by an Extended Kalman Filter Gadre and Stilwell (2005); Casey et al. (2007); Lee et al. (2007). The stochastic approach is not desirable when a guaranteed state estimation is needed as in the case of fault-tolerant control, Model Predictive approaches, or vehicle collision detection with obstacles.

Estimation for uncertain Linear Parameter-Varying (LPV) has mostly considered polytopes such as in Silvestre et al. (2017b). In the case of LTVs in discrete-time, there are proposals using intervals Thabet et al. (2014), zonotopes Combastel (2003) and ellipsoids Chernousko (2005) which are not accurate since intersections cannot be expressed in closed-form. Moreover, there are also approached resorting to ellipsotopes Kousik et al. (2022) and AH-polytopes Sadraddini and Tedrake (2019), even though these are not tailored for uncertain systems given the absence in the literature of explicit convex hull formulas. Techniques resorting to polytopes Silvestre et al. (2017a) and in the format of constrained zonotopes Scott et al. (2016) are the relevant techniques, although uncertainties inherently point towards computing a convex hull over the trajectories for vertices of the uncertain dynamics matrix. Please note that the uncertainties can also be modeled as exogenous signals at the expenses of a larger conservatism when the uncertainty matrices do not have rank equal to the

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unity Silvestre (2022b). The case of uncertainties cannot be addressed even considering the equivalent estimation tools for nonlinear systems in Abdallah et al. (2008), Alamo et al. (2005), Julius and Pappas (2009), Rego et al. (2018), Wan et al. (2018), respectively.

The recent work in Raghuraman and Koeln (2022) has introduced various set operations using constrained zonotopes and zonotopes. Among them is a closed-form expression for the convex hull of constrained zonotopes, albeit at the expenses of a large growth on the number of generator variables of $3(n_1+n_2)+1$ if the two original sets have n_1 and n_2 variables, respectively. This is quite far from the optimal for polytopes in explicit format of n_1+n_2+1. In this paper, we provide a closed-form description with n_1+n_2+1 variables (and also a linear growth in the number of constraints) for CCGs, which naturally extends to constrained zonotopes as these are a particular instance of CCGs. The main contributions can be highlighted as:

- Introduction of a closed-form expression that is exact for both polytopes (in constrained zonotope format) and CCGs that has the same complexity as the Minkowski sum (i.e., n_1+n_2 generator variables);
- The proposed method removes the growth factor associated with the convex hull, meaning that fewer order reduction procedures are required to maintain a tractable representation of the set-valued estimates.

The remainder of the paper is organized as follows. Section 2 formalizes the state estimation problem, highlighting the exponential growth of the auxiliary variables. We review in Section 3 the definition and main set operations for CCGs, while Section 4 is dedicated to presenting the proposed convex hull algorithm. Simulations using a unicycle model for a land autonomous vehicle are provided in Section 5. Conclusions and directions of future work are given in Section 6.

Notation : We let 0_n denote the n-dimensional vector of zeros and I_n the identity matrix of size n. The operator $\text{diag}(v)$ creates a diagonal matrix with v in the diagonal or extracts the diagonal if the argument is a matrix. The transpose of a vector v is denoted by $v^\intercal$, while the Euclidean norm for vector x is represented as $\|x\|_2 := \sqrt{x^\intercal x}$. On the other hand, $\|x\|_\infty := \max_i |x_i|$. The cartesian product is denoted by $\times$, the Minkowski sum of two sets by $\oplus$ and the intersection after applying a matrix R to the first set by $\cap_R$.

2. PROBLEM STATEMENT

The problem of state estimation in uncertain LPVs can be cast as finding a set of possible values given the measurements, disturbance, noise and initial state bounds and the model is given by:

$$\begin{aligned} x(k+1) &= \Big(F(\rho(k)) + \sum_{\ell=1}^{n_\Delta} \Delta_\ell(k) U_\ell\Big) x(k) + B(\rho(k))u(k) \\ &\quad + L(\rho(k))d(k) \\ y(k) &= C(\rho(k))x(k) + N(\rho(k))w(k) \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^{n_u}$, $d(k) \in \mathbb{R}^{n_d}$, $y(k) \in \mathbb{R}^m$ and $w(k) \in \mathbb{R}^{n_w}$ are the system state, input, disturbance signal, output and noise, respectively. The parameter $\rho(k)$ is the part of the parameters that can be measured at time k, which can be treated as in the case of LTVs. The main challenge appears from considering the n_Δ uncertainties denoted by Δ_ℓ and the constant matrices U_ℓ that account for how the uncertainties affect the nominal dynamics matrix given by $F(\rho(k))$. To lighten the notation, we will consider $F_k := F(\rho(k))$ and similarly for all the remaining matrices in (1). Notice that we have to explicitly consider ρ to account for nonlinearities that enter the model in a linear fashion as will happen with unicycle model used in Section 5. Moreover, in order to have a well-posed problem, we assume that all unknown signals are bounded within a compact convex set denoted by the correspondent capital letter, i.e., $x(0) \in X(0)$, $d(k) \in D(k)$ and $w(k) \in W(k)$. Without loss of generality, we will assume that $\forall k, |\Delta_\ell(k)| \leq 1$.

The problem addressed in this paper is summarized as:

Problem 1. Given compact convex sets $X(0)$, $D(k)$ and $W(k)$ for all $k \geq 0$ and measurements $y(k)$, how to compute a set $X(k)$ such that it is guaranteed that $x(k) \in X(k), \forall k \geq 0$.

Notice that Problem 1 is called *state estimation* although a converse definition could be presented for the output of the system (this is of particular interest in sensitivity analysis Silvestre et al. (2019) and system distinguishability Silvestre et al. (2021)). Problem 1 is quite general in terms of the measurement set $Y(k)$, i.e., the set of all state values that conform with the measurements $y(k)$. If there is range information, $Y(k)$ is an ellipsoid; in case of bearing angles, one would get $Y(k)$ to be a convex cone; and, if we have some norm-based measurement, $Y(k)$ is an affine transformation of an ℓ_p unit ball.

3. CONSTRAINED CONVEX GENERATORS OVERVIEW

In this section, we first review the main set operations and introduce the novel expression for the convex hull of the union of two CCGs. Definition 1 and Definition 2 provide a formal description of CCGs and the required operations.

Definition 1. (Constrained Convex Generators). A Constrained Convex Generator (CCG) $\mathcal{Z} \subset \mathbb{R}^n$ is defined by the tuple $(G, c, A, b, \mathfrak{C})$ with $G \in \mathbb{R}^{n \times n_g}$, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{n_c \times n_g}$, $b \in \mathbb{R}^{n_c}$, and $\mathfrak{C} := \{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_{n_p}\}$ such that:

$$\mathcal{Z} = \{G\xi + c : A\xi = b, \xi \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_{n_p}\}.$$

Definition 2. Consider three Constrained Convex Generators (CCGs) as in Definition 1:

- $Z = (G_z, c_z, A_z, b_z, \mathfrak{C}_z) \subset \mathbb{R}^n$;
- $W = (G_w, c_w, A_w, b_w, \mathfrak{C}_w) \subset \mathbb{R}^n$;
- $Y = (G_y, c_y, A_y, b_y, \mathfrak{C}_y) \subset \mathbb{R}^m$;

and a matrix $R \in \mathbb{R}^{m \times n}$ and a vector $t \in \mathbb{R}^m$. The three set operations are defined as:

$$RZ + t = (RG_z, Rc_z + t, A_z, b_z, \mathfrak{C}_z)$$

$$Z \oplus W = \left([G_z \; G_w], c_z + c_w, \begin{bmatrix} A_z & 0 \\ 0 & A_w \end{bmatrix}, \begin{bmatrix} b_z \\ b_w \end{bmatrix}, \{\mathfrak{C}_z, \mathfrak{C}_w\} \right)$$

$$Z \cap_R Y = \left([G_z \; 0], c_z, \begin{bmatrix} A_z & 0 \\ 0 & A_y \\ RG_z & -G_y \end{bmatrix}, \begin{bmatrix} b_z \\ b_y \\ c_y - Rc_z \end{bmatrix}, \{\mathfrak{C}_z, \mathfrak{C}_y\} \right).$$

Computationally speaking, it is required to store which type of generator we are using for which entries of the vector of auxiliary variables ξ. We would like to point out that all the aforementioned set representations are subsets of CCGs, namely:

- an interval corresponds to $(G, c, [\,], [\,], \|\xi\|_\infty \leq 1)$, for a diagonal matrix G;
- a zonotope is given by $(G, c, [\,], [\,], \|\xi\|_\infty \leq 1)$;
- an ellipsoid is defined by $(G, c, [\,], [\,], \|\xi\|_2 \leq 1)$, for a square matrix G;

- a constrained zonotope or polytope is $(G, c, A, b, \|\xi\|_\infty \leq 1)$;
- a convex cone in $\mathbb{R}^n$ is $(G, c, [\,], [\,], \xi \geq 0)$;
- ellipsotopes are given by $(G, c, A, b, \|\xi\|_{p_1} \leq 1, \cdots, \|\xi\|_{p_m} \leq 1)$, for some $p_i > 0$, $1 \leq i \leq m$;
- AH-polytopes are given by $(G, c, [\,], [\,], A\xi \leq b)$.

4. STATE ESTIMATION FOR UNCERTAIN LPVS USING CONSTRAINED CONVEX GENERATORS (CCGS)

In this section, the state estimation strategy is presented using CCGs and introducing the necessary convex hull operation to deal with the uncertainties. The main issue arising from each of the uncertainty parameters Δ_ℓ in (1) is that a product appears of the set $[-1, 1]$ with the CCG $X(k)$ when computing the set $X(k+1)$. The alternative that is typically explored is to consider the polytopic set of dynamics matrices and perform the convex hull for each of the vertices corresponding to $[-1, 1]^{n_\Delta}$ where the power of a set is understood as the cartesian product taken n_Δ times. Therefore, the propagation of the previous estimate $X(k)$ using the state equation in (1) corresponds to the set $X_{\text{prop}}(k+1)$:

$$X_{\text{prop}}(k+1) = \text{cvxHull}\left(\bigcup_{\Delta \in \text{vertex}([-1,1]^{n_\Delta})} \left(F_k + \sum_{\ell=1}^{n_\Delta} \Delta_\ell(k) U_\ell\right) X(k)\right) + B_k u(k) \oplus L_k D(k),$$

where cvxHull computes the convex hull of the argument.

Using the measurement equation in (1) corresponds to an intersection with $Y(k+1)$ that has all possible state values that conform with $y(k+1)$, meaning an update on the estimates given as follows:

$$X(k+1) = X_{\text{prop}}(k+1) \cap_C Y(k+1).$$

4.1 Convex Hull for CCGs

Let us start by defining the convex hull of two sets:

$$\text{cvxHull}(Z_1, Z_2) := \big\{z : z = \lambda z_1 + (1-\lambda) z_2, \lambda \in [0,1], z1 \in Z_1, z_2 \in Z_2\big\}.$$

Let us introduce a specific instance of norm cones that are going to be used in the following result. For a norm unity ball $\mathfrak{C}$ defined as $\|\xi\|_p \leq 1$, let us associate with it the correspondent norm cone of order zero $\mathfrak{C}^{(0)}(\lambda, a, b) := \|\xi\|_p + w_0 \lambda \leq v_0$ with the initialization of the row vector w_0 and column vector λ as empty and scalar $v_0 = 1$. In the base case, we can omit the arguments with a slight abuse of notation. We can now define norm cones of higher order of this operation in a recursive manner $\mathfrak{C}^{(\tau)}(\lambda, a, b) := \|\xi\|_p + [a \;\, bw_{\tau-1}]\, \lambda \leq bv_{\tau-1}$, such that the generator variable $\lambda \in \mathbb{R}^\tau$.

We can now state the main theorem introducing the closed-form expression for the convex hull of two CCGs and the complexity of this representation.

Theorem 1. Consider two Constrained Convex Generators (CCGs) as in Definition 1:

- $X = (G_x, c_x, A_x, b_x, \mathfrak{C}_x^{(\tau_x)}) \subset \mathbb{R}^n$;
- $Y = (G_y, c_y, A_y, b_y, \mathfrak{C}_y^{(\tau_y)}) \subset \mathbb{R}^n$;

such that $A_x \in \mathbb{R}^{n_c^x \times n_g^x}$, $A_y \in \mathbb{R}^{n_c^y \times n_g^y}$, $\xi_x \in \mathfrak{C}_x^{(\tau_x)} \implies \alpha \xi_x \in \mathfrak{C}_x^{(\tau_x)}$, for $\alpha \in [0, 1]$ and similarly for $\mathfrak{C}_y^{(\tau_y)}$.

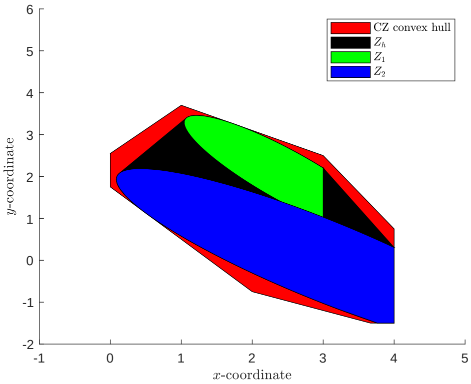

Fig. 1. Comparison between the set Z_h and the convex hull that one would obtain if first converted both Z_1 and Z_2 to constrained zonotopes by overbounding all convex generators by the ℓ_∞ unit ball.

The CCG corresponding to the exact convex hull $Z_h = (G_h, c_h, A_h, b_h, \mathfrak{C}_h) \subset \mathbb{R}^n$ is given by:

$$G_h = [G_x \;\, G_y \;\, c_x - c_y], c_h = \frac{c_x + c_y}{2},$$

$$A_h = \begin{bmatrix} A_x & 0 & -b_x \\ 0 & A_y & b_y \end{bmatrix}, b_h = \begin{bmatrix} \frac{1}{2} b_x \\ \frac{1}{2} b_y \end{bmatrix}$$

$$\mathfrak{C}_h = \{\mathfrak{C}_x^{(\tau_x+1)}(\lambda, -1, 0.5), \mathfrak{C}_y^{(\tau_y+1)}(\lambda, 1, 0.5), \mathbb{R}\},$$

which has $n_g^x + n_g^y + 1$ generators and $n_c^x + n_c^y$ constraints.

Proof. Following Theorem 1 from Conforti et al. (2020), we write Z_h as:

$$Z_h = \{p_h = G_x \xi_x + \lambda c_x + G_y \xi_y + (1-\lambda) c_y : 0 \leq \lambda \leq 1, A_x \xi_x \leq \lambda b_x, A_y \xi_y \leq (1-\lambda) b_y, \|\xi_x\|_{\ell_x} \leq \lambda, \|\xi_y\|_{\ell_y} \leq (1-\lambda)\}$$

when in the presence of unit balls.

By performing the substitution $\xi_\lambda = \lambda - 0.5$, we obtain a generator variable that belongs to the interval $[-0.5, 0.5]$ and after reorganizing to write everything in terms of $\xi_h = [\xi_x^\intercal \;\, \xi_y^\intercal \;\, \xi_\lambda]^\intercal$, we obtain:

$$Z_h = \{p_h = G_h \xi_h + c_h : A_h \xi_h \leq b_h, \|\xi_x\|_{\ell_x} \leq 0.5 + \xi_\lambda, \|\xi_y\|_{\ell_y} \leq 0.5 - \xi_\lambda\}.$$

where the norm cones correspond to $\mathfrak{C}_x^{(1)}(\xi_\lambda, -1, 0.5)$ and $\mathfrak{C}_y^{(1)}(\xi_\lambda, 1, 0.5)$. If on the other hand, we have a norm cones of order tau_x and τ_y, respectively, we obtain the expression $\mathfrak{C}_x^{(\tau_x+1)}(\xi_\lambda, -1, 0.5)$ and $\mathfrak{C}_y^{(\tau_y 1)}(\xi_\lambda, 1, 0.5)$. The number of generators and constraints comes directly from the size of the matrix A_h, which concludes the proof. ∎

Theorem 1 is not the exact convex hull since the last inequalities added were relaxed with the use of residual variables for a general convex generator. Figure 1 depicts an example of sets Z_1 and Z_2 with the respective set Z_h as given by Proposition 1 and what one would get if first converted the sets to constrained zonotopes and then applied the exact convex hull given in Raghuraman and Koeln (2022). As observed, even though the proposed method in Proposition 1 is not exact for CCGs, it still offers a better accuracy than computing the exact convex hull of the polytopic over-approximation of the sets.

The convex hull operator increases linearly the number of auxiliary variables to $n_g^x + n_g^y + 1$, however, this step has to be performed for all vertices which are exponential in the number of uncertainties. Such an issue was already present in Silvestre et al. (2017b) for polytopic set descriptions using the optimal convex hull formulation.

In order to keep the computation time for each iteration bounded, we introduce the order reduction in Algorithm 1, which computes a CCG with a specified number of constraints γ using $n+\gamma$ generators which is of the form of a polytope. The procedure starts by constructing a collections hyperplanes tangent to the surface in order to have a bounding polytope $v^\intercal x \leq b$, which is then converted to the CCG representation. We remark that if the CCG is representing a polytope (i.e., it is equivalent to a CZ) and vectors in v are all orthogonal to the facets of the polytope, then $X_{\text{red}}(k) = X(k)$ but with a decreased order in the representation. This is a trivial observation from the fact that $v'x \leq b$ would be the exact polytope. The min and max operations are element-wise.

Algorithm 1 Order Reduction using points on the surface.

Require: Set $X(K) \subseteq \mathbb{R}^n$ and desired order γ.
Ensure: Calculation of $X(k) \subseteq X_{\text{red}}(k) \subseteq \mathbb{R}^n$ with $n_g = \gamma + n$ generators and $n_c = \gamma$ constraints.

/* *Get points p_i on the surface such that $p_i = \arg\max v_i^\intercal p_i$, $1 \leq i \leq \gamma$* */
$[v, p] = \text{sampleSurface}(X(k), \gamma)$
/* *Compute box $\tilde{Z}$ for the points p* */
$\tilde{Z} = (\frac{1}{2}\text{diag}(\max p - \min p), \frac{1}{2}(\max p + \min p), [\,], [\,], \|\tilde{\xi}\|_\infty \leq 1)$
/* *Calculate b and σ such that all entries $v_i^\intercal p_i \in [\sigma, b]$*/
$\sigma = \min v^\intercal p$
$b = \text{diag}(v^\intercal p)$

$$X_{\text{red}}(k) = (\begin{bmatrix}\tilde{Z}.G & 0_{n\times\gamma}\end{bmatrix}, \tilde{Z}.c, \begin{bmatrix} v^\intercal \tilde{Z}.G & \frac{1}{2}\text{diag}(\sigma - b)\end{bmatrix}, \frac{b+\sigma}{2} - v^\intercal \tilde{Z}.c, \|\tilde{\xi}\|_{\text{inf}} \leq 1)$$

5. SIMULATIONS

In this section, simulations results are presented for a unicycle model of an autonomous vehicle in discrete-time for which there is a digital compass as an onboard sensor providing measurements of the orientation angle with a $\pm 5°$ error. Simulations were run in Matlab R2018a running on a HP machine with a Intel Core i7-8550U CPU @ 1.80GHz and 12 GB of memory resorting to Yalmip as the language to model optimization problems and Mosek as the underlying solver. Videos, figures and code can be found in `https://github.com/danielmsilvestre/CCGExactConvexHull`

We recover the example considering unicycle dynamics described in Hernández-Mendoza et al. (2011). The vehicle schematic representation is given in Figure 2 and has the following dynamics in discrete-time:

$$\begin{bmatrix} p_i \\ q_i \end{bmatrix}(k+1) = \begin{bmatrix} p_i \\ q_i \end{bmatrix}(k) + \text{Ts}\ A_i(\theta_i)\begin{bmatrix} v_i \\ w_i \end{bmatrix}(k)$$

where the state (p_i, q_i) identify the position of the front of the ith vehicle and the inputs (v_i, w_i) account for the linear velocity and rotation. Moreover, $\text{Ts} = 0.1$ stands for the sampling time, θ_i (we omit the time dependence in k for a more compact presentation) for the orientation and matrix $A_i(\theta_i)$ is given as:

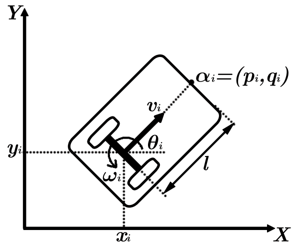

Fig. 2. Schematic of the unicycle model for the vehicles.

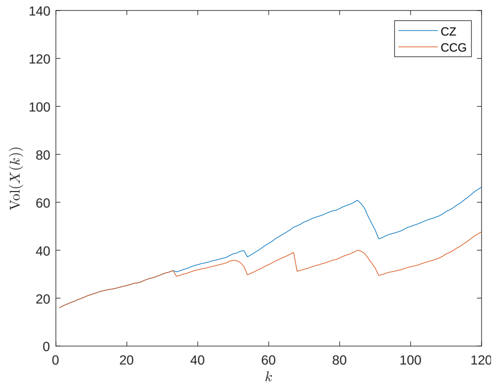

Fig. 3. Comparison of the volume for both set-valued estimates when using constrained zonotopes (CZ) and CCGs for the figure 8 trajectory.

$$A_i(\theta_i) = \begin{bmatrix} \cos\theta_i & -l\sin\theta_i \\ \sin\theta_i & l\cos\theta_i \end{bmatrix}.$$

In this simulation, we consider a single vehicle running for a total of 15 seconds and, assuming that the compass takes measurements $\hat{\theta_1}$ of the true variable θ_1 that have a maximum of $\pm 5°$ following a uniform distribution. Therefore, at each iteration time k, matrix A_1 in the dynamics is not available to the observer and we have to consider $\hat{\theta_1}$ to generate the nominal dynamics and an uncertainty Δ_1 with maximum magnitude of $5°$, which fits (1).

The trajectory-tracking control law used is:

$$\begin{bmatrix} v_i(k) \\ w_i(k) \end{bmatrix} = \frac{A_i^{-1}(\theta_i)}{\text{Ts}}\left(\tau(k+1) - \frac{\tau(k)}{2} - 0.5\begin{bmatrix} p_i(k) \\ q_i(k) \end{bmatrix} + d(k)\right)$$

where $\tau(k)$ accounts for the discrete sequence of waypoints in the trajectory. Once again, we assume that there is a telemetry sensor that produces estimates corrupted by noise of the value of $p_1(k)$ and $q_1(k)$ and add the corresponding disturbance term $d(k)$ to account for those differences. Moreover, there are two beacons at positions $[5\ 25]^\intercal$ and $[23\ 10]^\intercal$ that can be detected within a 5 and 2 units of distance which allows to better localize the vehicle.

The vehicle performs a figure 8 trajectory such that it can only get measurements from each beacon in one time interval. Figure 3 illustrates the volume evolution for the set-valued estimates $X(k)$ when using constrained zonotopes Scott et al. (2016) and CCGs when both used the same order reduction method in Section 4. Since the vehicle is moving and most of the time performing dead reckoning with the uncertain LPV model, the volume keeps increasing and is lowered when the vehicle reaches the beacon areas. The main trend to observe is that the added accuracy of the ℓ_2 ball representing the range measurement from the beacon contributes to a better performance of the CCG filter.

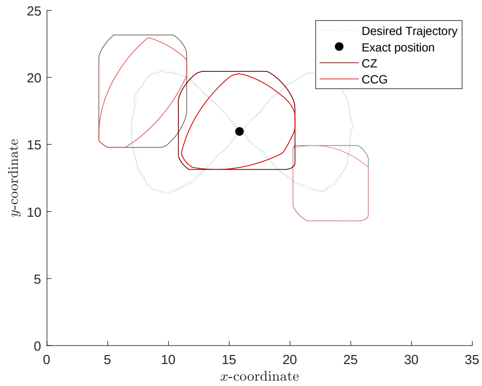

Fig. 4. Trajectory executed by the vehicle and the correspondent set-valued estimates at multiples of 40 iterations when using constrained zonotopes (CZ) and CCGs for the figure 8 trajectory.

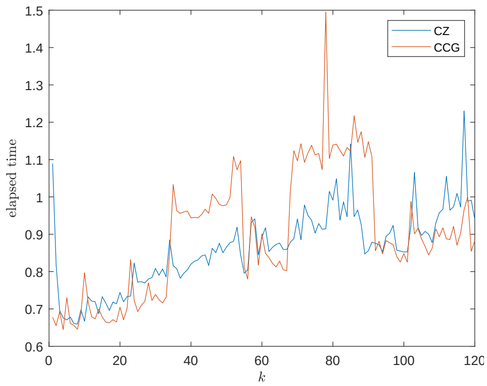

Fig. 5. Elapsed time for each iteration of both methods taking into account the constructing of the set, approximation algorithm and volume computation.

In Figure 4, it is illustrated the trajectory executed by the vehicle and the corresponding set-valued estimates using both the CZ and CCG approaches. We have selected a small number of time instants to display the sets as to avoid cluttering the image, but the full video can be found in the GitHub repository associated with the paper.

A last relevant issue is the elapsed time in each iteration taken by both filters with different set representations. Figure 5 shows the computation times across iterations during the whole simulation. At the beginning, both filters have very similar behavior pointing out to the fact that the CCG is yet to have round facets and the order reduction produces equivalent representations. However, as the simulation progresses the set is intersected with the range measurements. The curved boundaries of the CCGs result in a more complex representation. When the vehicle finds the second beacon and the set is considerably reduced in size, the CZ approach has a better performance given that $X(k)$ has a shape close to an interval, where its accuracy is the worst. This result points out to the need to further develop order reduction methods for CCGs that can exploit the nature of the sets. This is not a trivial task given the requirement of computing an outer-approximation to maintain the guaranteed feature in the estimation using set-membership approaches.

In order to illustrate an example where both filters should be similar, we simulated a spiral trajectory and increased the range of the beacons in 5 meters each. In this case, the trajectory is not taking advantage of the two beacons.

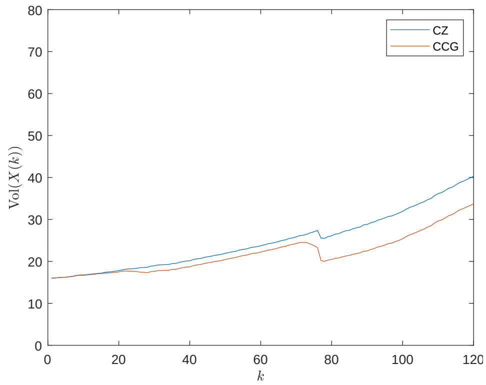

Fig. 6. Comparison of the volume for both set-valued estimates when using constrained zonotopes (CZ) and CCGs for the spiral trajectory.

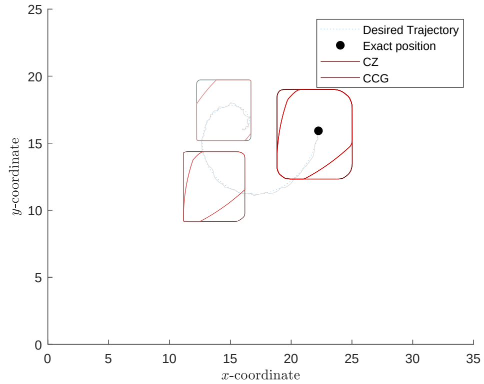

Fig. 7. Trajectory executed by the vehicle and the correspondent set-valued estimates at multiples of 40 iterations when using constrained zonotopes (CZ) and CCGs for the spiral trajectory.

However, the fact that the vehicle will receive the beacon more often should compensate. Figure 6 showcases that the volume is indeed much smaller for this trajectory since the vehicle performs dead reckoning less often. In this setup, the main difference between the two filters is precisely the representation of the circular shapes that benefits the CCGs.

In Figure 7, it is depicted the same snapshots for the trajectory where it is noticeable the rounded shapes corresponding to the range measurements. However, as seen in Figure 8, the more complicated set representation also reduces the performance of both filters. Similarly to the figure 8 trajectory scenario, both simulations illustrate a clear reduction in the conservatism without a very expressive increase in elapsed time for the overall computations. We remark that in terms of orders of magnitude, both filters in normal operation will take between 0.6 and 1.5 seconds, which is not viable for real-time applications and showcases the need to further purse efficient order reductions methods. We did not use the methods from CORA toolbox since we were obtaining even larger computing times since the Constrained Zonotopes format explodes in the number of variables.

6. CONCLUSIONS AND FUTURE WORK

In this paper, we have address the problem of set-valued estimation of autonomous vehicles with uncertainties in the dynamics. A direct example is the case of land robots that can be cast as uncertain Linear Parameter-Varying

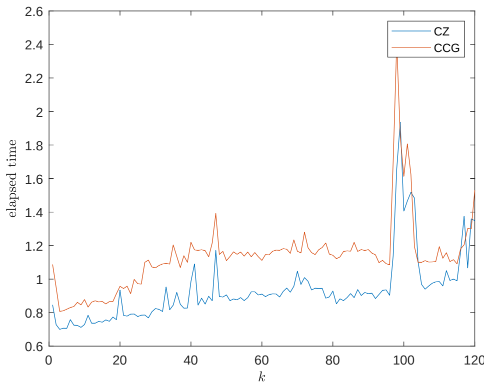

Fig. 8. Elapsed time for each iteration of both methods taking into account the constructing of the set, approximation algorithm and volume computation in the spiral trajectory scenario.

(LPV) models where the orientation angle being uncertain causes issues for techniques developed for LPVs. We then introduce a closed-form expression for convex hulls of Convex Generators (CCGs) that is optimal in terms of the number of generators and constraints since it combines both linearly. Such a result greatly impact the exponential growth when multiple uncertain parameters need to be used.

In a simulation representing a vehicle performing dead reckoning with occasional access to range measurements from beacons, it is shown that the current proposal significantly improves the estimation quality in comparison with Constrained Zonotopes that can hardly improve the set-valued estimates. As future work, increasing the performance of order reduction methods for CCGs that take into account the round nature of some of its facets can greatly improve the performance of the filter.

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