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The robust minimal controllability and observability problem

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Abstract

In this paper, we study the Robust Minimal Controllability and Observability Problem (rMCOP). The scenario that motivated this question is related to the design of a drone formation to execute some task, where the decision of which nodes to equip with a more expensive communication system represents a critical economic choice. Given a linear time-invariant system for each of the vehicles, this problem consists of identifying a minimal subset of state variables to be actuated and measured, ensuring that the overall formation model is both controllable and observable while tolerating a prescribed level of inputs/outputs that can fail. Based on the tools in the available literature, a naive approach would consist of enumerating separately all possible minimal solutions for the controllability and observability parts. Then, iterating over all combinations to find the maximum intersection of sensors/actuators in the independent solutions, yielding a combinatorial problem. The presented solution couples the design of both controllability and observability parts through a polynomial reformulation as a minimum set multi-covering problem under some mild assumptions. In this format, the algorithm has the following interesting attributes: (i) only requires the solution of a single covering problem; 9ii) using polynomial approximations algorithms, one can obtain close-to-optimal solutions to the rMCOP.

KEYWORDS

control applications, control design, minimal controllability and observability problem, robustness

1 | INTRODUCTION

Considering a Multi-Agent System (MAS) composed of vehicles interconnected by a communication network is a recurrent proposal for surveillance, exploration, and measuring tasks to be accomplished by unmanned and automatic robotic systems. Missions entailing the use of a large number of such vehicles can adopt a leader/followers approach[1,2] characterized by having: (i) expensive nodes (leaders) that can communicate with a ground station to receive mission commands and that might be equipped with sophisticated sensors or localization equipment: (ii) cheaper drones (followers) implementing local controllers based on onboard sensors that measure relative localization and receive a small amount of data from the leaders. In this scenario shown in Figure 1, a critical task is to minimize the number of leaders for economic reasons without compromising the controllability and observability of the overall system. In distributed environments, we should ensure these two critical properties.[3,4] We will refer to this challenge as the Minimal Controllability and Observability Problem (MCOP).

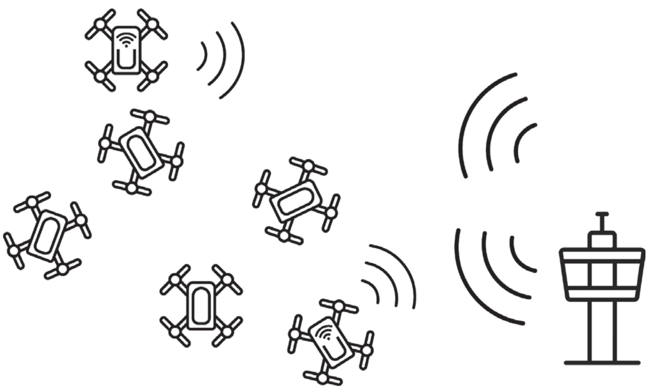

FIGURE 1 Depiction of the envisioned scenarios where the fixed tower represents the ground station, there are two expensive nodes (leaders) with the extra wireless symbol and four followers that use a controller based on local information related to nearby leaders

In this paper, we focus on the subset of systems where the nodes in the MCOP can be described by Linear Time-Invariant (LTI) models, which can be the result of a linearization of the original nonlinear model. Given that in a realistic environment there can be actuators or sensors that stop working either due to hardware faults, natural *phenomena* (e.g., due to the adverse nature of the environments where the actuators and sensors are placed) or by some software malfunction caused by an external entity (e.g., the scenario that took place in the Stuxnet malware incident[5]), we also study the robust version. In the case of a malicious entity, we assume that there is an external procedure for the identification/detection of attacked sensors/actuators, such as References 6 or 7. Our goal is to devise a mechanism to ensure that the system can recover, resorting to other sensors and actuators, ensuring the system controllability and observability. In this case, the goal is to select a minimal number of leaders such that even if we have a prescribed number of input and output failures, the system is still controllable and observable, and we will refer it as the Robust Minimal Controllability and Observability Problem (rMCOP). In other words, if a prescribed number of inputs and outputs fail, there is redundancy on the number of inputs and outputs that ensures the system to still be controllable and observable.

1.1 | Related work

The controllability aspect of a dynamical system is dual to the observability of linear systems.[8] In particular, MAS emerge in a plethora of areas, such as mathematics, biology, physics, sociology, and engineering applications,[9-14] and can be often represented by a Linear Time-Invariant (LTI) or a Linear Parameter-Varying (LPV) system. The controllability of MAS having Laplacian dynamics was initially investigated by Tanner.[15] Rahmani el al.[16] and Egerstedt et al.[17] found necessary and sufficient conditions in terms of partitions of the Laplacian graph for controllability. Paths and cycles were investigated by Parlangeli et al.[18] and then extended to the controllability of grid graphs via reductions, symmetries, and scaled operations on the Laplacians.[19] Tian et al.[20] studied controllability and observability of MAS with heterogeneous and switching topologies, where the model equations for the position and velocity are different and switch in the network. Later, Tian et al.[21] studied the controllability and observability of switched multi-agent systems (MAS) by constructing a switching sequence that ensures controllability, resorting to the concepts of the invariant subspace and the controllable state set. Necessary and sufficient conditions for both controllability and observability are also presented. More recently, Ramos et al.[22] presented a framework to study the resistance to bribery of nodes in a network, using control ideas, via average consensus.

Structural systems[23] allow us to efficiently design a minimal input or output placement for classes of LTI systems, by exploring the pattern of zeros and nonzeros of the dynamics matrix of the system. These are powerful tools for efficient design, ensuring almost surely the controllability (or observability) of the underlying system. Therefore, several frameworks have been developed in this scope. Pequito et al.[24] presented an efficient and unified framework to select the minimum number of manipulated/measured variables to reach the structural controllability/observability of the system. Also, it is provided a method to select the minimum number of feedback interconnections between measured and manipulated variables, ensuring the closed-loop system has no structural fixed modes. A model checking framework for LTI switching systems, using structural systems analysis, was presented by Ramos et al.[25] and used by Ramos et al.[26] to do the analysis and design of electric power grids with p-robustness guarantees, ensuring structural controllability (that is, guaranteeing resilience to at most p transmission lines failures). However, the notions of structural controllability and structural observability are necessary, but not sufficient conditions to ensure systems' controllability and observability. These structural notions can be used when the parameters of the dynamics matrix are independent. In general, this requirement is not valid when considering MASs. In this case, we can find block-diagonal dependencies of parameters that, for instance, model two agents with the same dynamics.

Solving the Minimal Controllability Problem (MCP) was shown to be NP-hard by Olshevsky.[27] Pequito et al.[28] extended the MCP to address the robustness to inputs that may fail over time, showing that the Robust Minimal Controllability Problem (rMCP) is equivalent to a minimum set multicovering problem, for which there exists efficient approximation algorithms with close-to-optimal guarantees. The authors extended the results for switched linear continuous-time systems in Reference 29.

The previously proposed frameworks to address the MCP (or rMCP) can be used, by duality, to address the Minimal Observability Problem (MOP) (or its robust version rMOP). Therefore, we can design the input and output matrices by solving two independent problems. In contrast, in this paper, we want to identify a minimal subset of state variables that, when actuated and controlled, the system is controllable and observable, the MCOP problem. First, we notice that solving the MCP and the MOP independently does not produce, in general, a minimal solution to the MCOP. A brute-force possibility to solve the MCOP can be accomplished by enumerating all possible solutions to the MCP and all possible solutions to the MOP, and to select a pair of solutions with a maximal intersection. Notwithstanding, it would require, in general, a prohibitive computational effort.

For the first time, we address the problem of finding a minimal solution that ensures that the system is controllable and observable by affecting and measuring the smallest number of agent states. In the remainder of this manuscript, we show that solving the MCOP is equivalent to another instance of the minimum set multicovering problem.

Main contributions of this paper are the following: (i) we reduce the MCOP and the rMCOP to minimum set multicovering problems; (ii) we show that almost all numerical instances of the input and output matrices that satisfy a specified structure yield solutions to the MCOP and rMCOP; (iii) we present a method that, given the structure of the input and the output matrices, produces numerical instances which are solutions to the rMCOP; (iv) we present illustrative examples of the main results.

The rest of the paper has the following structure. In Section 2, we present a formal description of the MCOP and rMCOP. Section 3 sets up the notation adopted in this work and the preliminary definitions that are basilar to the main results, which are presented in Section 4. In Section 5, we illustrate the main result of the manuscript, and we close the paper in Section 6, drawing conclusions and future research directions.

2 | PROBLEMS STATEMENT

An LTI system, under the adversarial scenario of failure or a malicious entity tempering with system inputs and/or outputs and identified by an external procedure, may be described as

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B^{\mathcal{M}\setminus\mathcal{A}}u(t)\\ y(t) &= C^{\mathcal{N}\setminus\mathcal{A}}x(t),\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state of the system, $x(0) = x_0$, $u(t) \in \mathbb{R}^p$ is a continuous input signal and $y(t) \in \mathbb{R}^q$ is a continuous output signal. Furthermore, $B^{\mathcal{M}\setminus\mathcal{A}}$ denotes the set of columns of B with indices in $\mathcal{M} \setminus \mathcal{A}$, where $\mathcal{M} = \{1, \dots, p\}$ is the set of input indices and $\mathcal{A}$ the set of indices of malfunctioning inputs/outputs. Analogously, $C^{\mathcal{N}\setminus\mathcal{A}}$ is the set of rows of C with indices in $\mathcal{N} \setminus \mathcal{A}$, where $\mathcal{N} = \{1, \dots, q\}$ is the set of output indices.

For convenience, let us refer to a system given in the format of (1) by the triple $(A, B^{\mathcal{M}\setminus\mathcal{A}}, C^{\mathcal{N}\setminus\mathcal{A}})$ and, when $\mathcal{A}$ is explicit from the context, we simply use the triple (A, B, C), to simplify the notation.

Usually, in the multi-agent scenario, matrix A has blocks in the diagonal containing each agent's dynamics, and the remaining entries encode how the state of one agent influences a neighbor depending on the distributed algorithm being employed at the local level. For instance, if the agents are performing a consensus on the velocity and position such that interagent distances are equal to some prescribed values to enforce a formation moving with the same velocity, matrix A off-diagonal blocks will be doubly stochastic matrices (assuming network edges are bidirectional). In this context, matrices B and C translate which nodes receive control actuation from and send measurements to some entity driving the formation. For instance, a human can increase the formation velocity by sending a message to node i. In that case, only the ith block of matrix B will have nonzero entries. Therefore, nonzero entries correspond to equipping the correspondent node with more costly communication devices, which preferably implement a bidirectional communication (i.e., having capabilities to sense and actuate). This scenario motivates the need to guarantee controllability and observability of the whole MAS while reducing the number of nonzero entries for different nodes. The proposed cost function translates this idea by favoring solutions that assign nonzeros to the minimum possible number of agents, that is, where we only need a small number of communicating device nodes. We illustrate the proposed method with a MAS scenario in the example provided in Section 5.4, where we compare our approach with the naïve strategy. Notwithstanding, in this paper, the proposed approach is general and applies to any matrices A, B, and C.

The first problem that we tackle is how to find two minimal sets—one for the states to be actuated and another for states to be measured—such that the intersection has maximum cardinality, while keeping the overall system controllable and observable. Let $\|\cdot\|_0$ denote the semi-norm function that counts the number of nonzero entries of a vector or a matrix. Additionally, let M_1, M_2 be two $n_1 \times n_2$ matrices. We define the matrix $M_3 = M_1 \vee M_2$ to be the $n_1 \times n_2$ matrix such that the entry $[M_3]_{i,j} = 0$ if $[M_1]_{i,j} = [M_2]_{i,j} = 0$, and $[M_3]_{i,j} = 1$, otherwise.

Therefore, a cost function that combines matrices B and C can model, for example, a MAS composed of quad-rotors drones with an established communication network. Additionally, some of these quad-rotors may need to be more robust (and therefore more expensive) to be equipped with communication antennas to communicate (control and measure their state) with a central entity, and equipped with heavier batteries. Hence, it is important to minimize the number of more expensive quad-rotors, which is exactly encoded in the proposed cost function. The problem can be stated as the solution to the following optimization problem:

Problem 1 (MCOP). Given a LTI system $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$, $x(0) = x_0$ and $x \in \mathbb{R}^n$, design $B \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{n\times n}$ as a solution of the optimization problem:

$$\begin{aligned}(B^*, C^*) \quad &= \quad \underset{B,C\in\mathbb{R}^{n\times n}}{\arg\min} \|B \vee C\|_0, \\ \text{s.t.} \quad & (A, B, C) \text{ is controllable and observable.}\end{aligned}$$

Observe that we allow the matrices B and C to have sizes $n \times n$ to ensure that a solution always exists, that is, picking $n \times n$ identity matrices ensures controllability and observability. The second problem addressed in this paper is the robust version of Problem 1. In other words, besides ensuring the system to be controllable and observable, we further want to guarantee the system to remain controllable and observable despite the existence of input or output failures.

Problem 2 (rMCOP) 2. Given a LTI system $\dot{x}(t) = Ax(t) + B^{\mathcal{M}\setminus\mathcal{A}}u(t)$, $y(t) = C^{\mathcal{N}\setminus\mathcal{A}}x(t)$, $x(0) = x_0$ with $x \in \mathbb{R}^n$, and given the maximum number of inputs+outputs that may fail, $s \in \mathbb{N}$, find $B, C^\intercal \in \mathbb{R}^{n\times(s+1)n}$ as a solution of the optimization problem:

$$\begin{aligned}(B^*, C^*) = \quad & \underset{B,C^\intercal\in\mathbb{R}^{n\times(s+1)n}}{\arg\min} \|B \vee C\|_0, \\ \text{s.t.} \quad & (A, B^{\mathcal{M}\setminus\mathcal{A}}, C^{\mathcal{N}\setminus\mathcal{A}}) \text{ is controllable and observable} \\ & \mathcal{A} \subset \mathcal{M} \cup \mathcal{N} \text{ and } |\mathcal{A}| \le s\end{aligned}$$

Notice that, in this case, we allow the matrices B and C to have sizes $n \times (s+1)n$ to ensure that a solution always exists, that is, the concatenation of $(s+1)$ times the $n \times n$ identity matrix always ensure that the system is controllable and observable even when s inputs+outputs fail.

Remark that Problem 1 is a particular case of Problem 2 where $s = 0$. In the remainder, we focus on Problem 2 and get as a byproduct the solution to Problem 1.

The following assumptions are needed for the adopted solution.

Assumption 1. The dynamics matrix A has simple eigenvalues, that is, $\sigma(A) = \{\sigma_1, \dots, \sigma_n\}$ and $\sigma_i \neq \sigma_j$ for all $i \neq j$.

Note that Assumption 1 is common[28,29] and not very restrictive, because there are several applications where this assumption holds. Namely, dynamical systems that are modeled by random networks of the Erdös-Rényi type;[30] and popular benchmark dynamical systems utilized in control systems engineering.[31,32]

Assumption 2. A left-eigenbasis and a right-eigenbasis of A are available, that is, we have access to the left-eigenvectors and right-eigenvectors.

Assumption 2 is technically needed although, in practice, we may only compute the eigenvectors up to a certain precision.

3 | PRELIMINARIES AND TERMINOLOGY

We denote the $n \times n$ identity matrix by I_n. We denote sets of numbers by calligraphic letters, for example, $\mathcal{I}, \mathcal{S}, \mathcal{U}, \mathcal{J}$. We denote by $I_n(\mathcal{I})$, where $\mathcal{I} \subseteq \{1, \dots, n\}$, the $n \times n$ matrix with the columns with indices in $\mathcal{I}$ equal to the columns of I_n and the remaining columns equal to zero. Analogously, given a matrix $B \in \mathbb{R}^{n\times m}$ and a set $\mathcal{M} \subset \{1, \dots, m\}$, we denote by $B(\mathcal{M}) \in \mathbb{R}^{n\times m}$ the matrix composed by the columns of B with indices in $\mathcal{M}$ and the remaining columns equal to zero. We denote by $\mathbf{0}_{n,m}$ the $n \times m$ matrix of zeros and, similarly, by $\mathbf{1}_{n,m}$ the $n \times m$ matrix of ones. Further, when the dimensions are obvious from the context, we omit the dimensions and write $\mathbf{0}$ and $\mathbf{1}$. Given a square matrix A we denote its *spectrum* by $\sigma(A)$, that is, the set of eigenvalues of A.

We will use the Popov–Belevitch–Hautus (PBH) eigenvector controllability and observability tests. Consider an LTI system $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$ and $x(0) = x_0$, with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$ and $y(t) \in \mathbb{R}^q$. The PBH eigenvector test for controllability states that the system is controllable if $v^\intercal B \neq 0$ for each left-eigenvector v of A. Similarly, the PBH eigenvector test for observability states that the system is observable if $Cu \neq 0$ for each right-eigenvector u of A.

Definition 1 (Minimum Set Covering Problem[33]). Given a universe of m elements $\mathcal{U}$, a collection of n set $\{\mathcal{S}_1, \dots, \mathcal{S}_n\}$, with $\mathcal{S}_i \subseteq \mathcal{U}$, for $i \in \{1 \dots, n\}$, such that $\bigcup_{i=1}^{n} \mathcal{S}_i = \mathcal{U}$. The minimum set covering problem consists in finding a set of indices $\mathcal{J}^* \subseteq \{1, \dots, n\}$ such that $\bigcup_{i\in\mathcal{J}^*} \mathcal{S}_i = \mathcal{U}$. That is,

$$\mathcal{J}^* = \underset{\mathcal{J}\subseteq\{1,\dots,n\}}{\arg\min} |\mathcal{J}|. \tag{2}$$

A generalization of the previous problem consists in requiring each element of the universe to be covered, at least, a specified number of times. This extension has the following definition.

Definition 2 (Minimum Set Multicovering Problem[34]). Given a universe of m elements $\mathcal{U}$, a collection of n sets $\{\mathcal{S}_1, \dots, \mathcal{S}_n\}$, with $\mathcal{S}_i \subseteq \mathcal{U}$, for $i \in \{1 \dots, n\}$, such that $\bigcup_{i=1}^{n} \mathcal{S}_i = \mathcal{U}$, and a demand function $d : \mathcal{U} \to \mathbb{N}$ indicating the number of times an element u has to be covered. In other words, $d(u)$ is the minimum number of sets that containing element u that need to be considered. The minimum set multicovering problem consists in finding a set of indices $\mathcal{J}^* \subseteq \{1, \dots, n\}$ such that $\bigcup_{i\in\mathcal{J}^*} \mathcal{S}_i = \mathcal{U}$ and every element $u \in \mathcal{U}$ is covered $d(u)$ times. That is,

$$\begin{aligned} \mathcal{J}^* = &\underset{\mathcal{J}\subseteq\{1,\dots,n\}}{\arg\min} |\mathcal{J}| \\ &\text{s.t. } |\{i \in \mathcal{J} : u \in \mathcal{S}_i\}| \geq d(u). \end{aligned} \tag{3}$$

The previous problems will play a key role in our proposal for Problems 1 and 2.

4 | MAIN RESULTS

In this section, we investigate solutions to Problems 1 and 2 by rewriting them as in minimum set multicovering problems. To that end, we present the following algorithm.

Algorithm 1. Polynomial reduction of the structural optimization Problem 2, to a set multicovering problem

Input: $\{\bar{v}^j\}_{j=1}^{n}$ and $\{\bar{u}^j\}_{j=1}^{n}$, two collections each of n vectors, both in $\{0, \star\}^n$ and $s \in \mathbb{N}$, the maximum number of inputs and outputs that may fail.
Output: $\mathcal{S} = \{\mathcal{S}_j\}_{j\in\{1,\dots,(s+1)n\}}$ and $\mathcal{U}$, a set with n sets, and the universe of these sets, respectively.

$\mathcal{S}_j = \emptyset$, for $j \in \{1, \dots, (s+1)n\}$
for $j = 1, \dots, n$ **for** $k = 1, \dots, n$ **if** $[\bar{v}^j]_k \neq 0$ **then** $\mathcal{S}_k = \mathcal{S}_k \cup \{j\}$
for $j = 1, \dots, n$ **for** $k = 1, \dots, n$ **if** $[\bar{u}^j]_k \neq 0$ **then** $\mathcal{S}_k = \mathcal{S}_k \cup \{n + j\}$
for $l = 1, \dots, s - 1$ **for** $k = 1, \dots, n$ $\mathcal{S}_{ln+k} = \mathcal{S}_k$
set $\mathcal{S} = \{\mathcal{S}_j\}_{j\in\{1,\dots,(s+1)n\}}$, $\mathcal{U} = \bigcup_{\mathcal{V}\in\mathcal{S}} \mathcal{V}$ and $d(i) = s + 1$ for $i \in \mathcal{U}$.

Notice that, in Algorithm 1, we use the same set $\mathcal{S}_k$ to place an element for both sets of vectors. Therefore, to distinguish from which of the sets of vectors an element of a set $\mathcal{S}_k$ is original, we need to verify if it is smaller or equal to n. Furthermore, notice that, since we are combining information from the set of left-eigenvectors and the set of right-eigenvectors, $\left\{\bar{v}^j\right\}_{j=1}^n$ and $\left\{\bar{u}^j\right\}_{j=1}^n$, in the same set $\mathcal{S}_k$, we are encoding the cost function of Problem 2 in the set multicovering problems. This combination forces the set multicovering problem solution to select indices of sets that preferably cover information from both left-eigenvectors and right-eigenvectors.

Lemma 1. *Given two collections of nonzero vectors* $\{\bar{v}^j\}_{j=1}^n$ *and* $\{\bar{u}^k\}_{k=1}^n$, *with* $\bar{v}^j, \bar{u}^k \in \{0,1\}^n$, *and* $s \in \mathbb{N}$, *finding* $\overline{B}^* \in \{0,1\}^{n\times n}$ *and* $\overline{C}^* \in \{0,1\}^{n\times n}$ *such that*

$$\begin{aligned} (\overline{B}^*, \overline{C}^*) \quad & = \arg\min_{\overline{B},\overline{C}\in\{0,1\}^{n\times n}} \|\overline{B} \vee \overline{C}\|_0 \\ \text{s.t.} \quad & \bar{v}^j \cdot \overline{B}(\mathcal{M}) \neq 0 \ \text{ and} \\ & \bar{u}^k \cdot \overline{C}(\mathcal{M}) \neq 0, \ \text{ for all } j,k \in \{1, \dots, n\}, \\ & \text{where } \ \mathcal{M} \subset \{1, \dots, n\} \ \text{ and } \ |\mathcal{M}| \geq n - s. \end{aligned} \tag{4}$$

is polynomially reducible, in n, to a minimum set multicovering problem with universe $\mathcal{U}$, *collection of sets* $\mathcal{S}$ *and demand function d by applying Algorithm 1.*

Proof. Let $\mathcal{S}$ be a collection of sets, $\mathcal{U}$ a universe and d a demand function that result from Algorithm 1. Then, we have the following equivalences. Let $\mathcal{I} \subset \{1, \dots, n, \dots, (s+1)n\}$ be a set of indices. Further, let $\mathcal{I}^b = \{i \ : \ i \in \mathcal{I} \ \text{ and } \ \exists j \ [\bar{v}^j]_i \neq 0\}$, and let $\overline{B} \equiv \overline{B}(\mathcal{I}^b) \in \{0,1\}^{n\times(s+1)n}$ be a structural matrix such that $[\overline{B}]_{j,i} \neq 0$, with $j = (i \bmod n) + 1$, if and only if $i \in \mathcal{I}^b$. Analogously, let $\mathcal{I}^c = \{i \ : \ i \in \mathcal{I} \ \text{ and } \ \exists j \ [\bar{u}^j]_i \neq 0\}$, and let $\overline{C} \equiv \overline{C}(\mathcal{I}^c) \in \{0,1\}^{n\times(s+1)n}$ be a structural matrix such that $[\overline{C}]_{j,i} \neq 0$, with $j = (i \bmod n) + 1$, if and only if $i \in \mathcal{I}^c$. Observe that $\mathcal{I} = \mathcal{I}^b \cup \mathcal{I}^c$. Then, since $\mathcal{I}$ is a solution of the set multicovering problem, we have that the collection of sets $\{\mathcal{S}_i\}_{i\in\mathcal{I}}$ covers $\mathcal{U}$ and satisfies the demand function d if and only if $\forall j \in \{1, \dots, n\} \ \exists l \in \mathcal{I}$ such that $j \in \mathcal{S}_l$ and $|\{r \in \mathcal{I} \ : \ j \in \mathcal{S}_r\}| \geq d(j) = s+1$. This is equivalent to the conjunction of the twofold:

(*i*) $\forall j \in \{1, \dots, n\} \ \exists l \in \mathcal{I}^b$ such that $\bar{v}^j_l \overline{B}_l^\intercal \neq \mathbf{0}$ and $|\{r \ : \ \bar{v}^j_r \overline{B}_r^\intercal \neq \mathbf{0}\}| \geq d(j) = s+1$;
(*ii*) $\forall k \in \{1, \dots, n\} \ \exists l \in \mathcal{I}^c$ such that $\bar{u}^k_l \overline{C}_l^\intercal \neq \mathbf{0}$ and $|\{r \ : \ \bar{u}^k_l \overline{C}_l^\intercal \neq \mathbf{0}\}| \geq d(k) = s+1$;

where $\bar{v}^j_r \overline{B}_r^\intercal$ (and, analogously, $\bar{u}^k_l \overline{C}_l^\intercal$) is the scalar product of rth the column of matrix $\overline{B}$ with the element $\bar{v}^j_r$. That is, $\overline{w} = \bar{u}^k_l \overline{C}_l^\intercal$ and $\overline{w}_i \neq 0$ if $[\overline{B}_r^\intercal]_i \neq 0$ and $\bar{v}^j_r \neq 0$ and $\overline{w}_i = 0$ otherwise. This is equivalent to $\forall j \in \{1, \dots, n\} \ \bar{v}^j \cdot \overline{B} \neq \mathbf{0}$ and $|\{l \ : \ v^j \cdot \overline{B}_l^\intercal \neq \mathbf{0}\}| \geq s+1$, and, similarly, $\forall k \in \{1, \dots, n\} \ \bar{u}^k \cdot \overline{C} \neq \mathbf{0}$ and $|\{l \ : \ u^k \cdot \overline{C}_l^\intercal \neq \mathbf{0}\}| \geq s+1$. Hence, even if s entries of $\overline{B}$ become zero (s inputs fail) it is still true that $\bar{v}^j \cdot \overline{B} \neq 0$. Analogously, if s entries of $\overline{B}$ become zero (s outputs fail) it is still true that $\bar{u}^k \cdot \overline{C} \neq \mathbf{0}$.

In summary, $(\overline{B}, \overline{C})$ is a feasible solution to Problem 2. Also, notice the selection of the minimal set of indices that correspond to sets that simultaneously cover left-eigenvectors and right-eigenvectors, corresponds to minimize the cost function of Problem 2. Moreover, we observe that the reduction of Problem 2 to a minimum set multicovering problem produces a sparsity pattern $(\overline{B}, \overline{C})$ that is the sparsity of a solution to Problem 2. Finally, the cost of Algorithm 1 is $\mathcal{O}(n^3)$ and, as envisaged, the aforementioned reduction has polynomial time and space complexity in n. ■

In fact, given the structure of a feasible solution of Problems 1 or 2, any numerical realization leads to a solution for the problem.

Now, building upon Lemma 1 , we state the main result of this paper.

Theorem 1. *The rMCOP can be solved in two steps:*

(i) identifying the sparsity of the input and output matrices, $(\overline{B}, \overline{C})$, *using Lemma 1;*
(ii) choosing a numerical realization for $(\overline{B}, \overline{C})$.

Proof. By Assumption 2 a left and a right-eigenbasis are available and by assumption 1 each eigenbasis is composed by n vectors. Hence, the proof follows by noticing that Lemma 1 produces a feasible solution to the rMCOP that satisfies the PHB eigenvector test for controllability and observability, that can be used to find a numerical solution. ■

Further, we observe that to solve a rMCOP (or a MCOP) is computationally demanding. In fact, we have the following.

Theorem 2. *Both the MCOP and the rMCOP are NP-hard.*

Proof. The proof follows by noticing that a subproblem of the MCOP is the minimal controllability problem (MCP) and a subproblem of the rMCOP is the robust minimal controllability problem (rMCP), that is, when we only seek to find an input matrix, and both problems are NP-hard,[28] that is when one of the input collections of the structural vectors in the optimization problem (4) (and in Algorithm 1) is empty. Additionally, the optimization problem (4) is equivalent to a set multicovering problem, as shown in the proof of Lemma 1. Hence, equivalent to an NP-hard problem. ■

Next, we illustrate the main results with examples.

5 | ILLUSTRATIVE EXAMPLES

In the following, we illustrate the use of the proposed methods with synthetic and real-world examples.

5.1 | Synthetic examples

Consider the linear system with dynamics matrix

$$A = \frac{1}{2}\begin{bmatrix} 10 & 0 & 0 & 2 & 2 \\ 3 & 6 & 3 & 2 & 1 \\ -3 & 0 & 3 & -4 & -3 \\ 3 & 0 & -1 & 6 & 3 \\ -3 & 0 & 1 & 2 & 5 \end{bmatrix}.$$

The eigenvalues of A are $\sigma = \{1, 2, 3, 4, 5\}$ and hence the matrix has simple eigenvalues. The left-eigenvectors of A are

$$V^L = \begin{bmatrix} | & | & | & | & | \\ v_1^L & v_2^L & v_3^L & v_4^L & v_5^L \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Further, the right-eigenvectors of A are

$$V^R = \begin{bmatrix} | & | & | & | & | \\ u_1^R & u_2^R & u_3^R & u_4^R & u_5^R \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} -2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

We start by addressing Problem 1. To illustrate the main results, we follow two approaches:

(*i*) find, independently, a minimal solution to ensure controllability and a minimal solution to ensure observability with the maximum number of common state variables to actuate and observe;
(*ii*) use Algorithm 1 to solve the MCOP (rMCOP with $s = 0$).

5.2 Approach (*i*)

If we use the method in Reference 28 to find a solution to the minimal controllably problem, we build the sets

$$\begin{array}{ll} \mathcal{S}_1^1 = \{1,4\} & \leftarrow \\ \mathcal{S}_2^1 = \{3\} & \leftarrow \\ \mathcal{S}_3^1 = \{3,4,5\} & \leftarrow \\ \mathcal{S}_4^1 = \{1,2,3,5\} & \leftarrow \\ \mathcal{S}_5^1 = \{1,2,3,4\} & \leftarrow \end{array} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} = V^L.$$

This sets constitute the universe $\mathcal{U} = \bigcup_{i=1}^{5} \mathcal{S}_i^1 = \{1, \ldots, 5\}$. By solving the associated minimum set covering problem, the possible solutions are $\mathcal{I}_1^1 = \{1,4\}$ ($\mathcal{U} = \mathcal{S}_1^1 \cup \mathcal{S}_4^1$), $\mathcal{I}_2^1 = \{3,4\}$ ($\mathcal{U} = \mathcal{S}_3^1 \cup \mathcal{S}_4^1$), $\mathcal{I}_3^1 = \{3,5\}$ ($\mathcal{U} = \mathcal{S}_3^1 \cup \mathcal{S}_5^1$) and $\mathcal{I}_4^1 = \{4,5\}$ ($\mathcal{U} = \mathcal{S}_4^1 \cup \mathcal{S}_5^1$).

Analogously, by invoking the duality between controllability and observability, we can use[28] to solve the dual problem, the minimal observability problem. In this case, we build the sets

$$\begin{array}{ll} \mathcal{S}_1^2 = \{1,2\} & \leftarrow \\ \mathcal{S}_2^2 = \{1,2,3,4,5\} & \leftarrow \\ \mathcal{S}_3^2 = \{1,4,5\} & \leftarrow \\ \mathcal{S}_4^2 = \{1,4,5\} & \leftarrow \\ \mathcal{S}_5^2 = \{1,2,4,5\} & \leftarrow \end{array} \begin{bmatrix} -2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix} = V^R.$$

Again, this sets constitute the universe $\mathcal{U} = \bigcup_{i=1}^{5} \mathcal{S}_i^2 = \{1, \ldots, 5\}$. Solving the associated minimum set covering problem, the possible solution is $\mathcal{I}_1^2 = \{2\}$. Therefore, the pairs of solutions to both problems with maximum intersection are: $(\mathcal{I}_1^1, \mathcal{I}_1^2)$, $(\mathcal{I}_1^1, \mathcal{I}_1^2)$, $(\mathcal{I}_2^1, \mathcal{I}_1^2)$, $(\mathcal{I}_3^1, \mathcal{I}_1^2)$ and $(\mathcal{I}_4^1, \mathcal{I}_1^2)$. All the cases result in input matrices and output matrices where $\|B\|_0 + \|C\|_0 - \|B \odot C\|_0 = 2 + 1 - 0 = 3$. For instance, for the pair of solutions $(\mathcal{I}_1^1, \mathcal{I}_1^2)$, we obtain the following

$$B(\mathcal{I}_1^1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$C(\mathcal{I}_1^2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Notice that $v_i^L \cdot B(\mathcal{I}_1^1) \neq 0$ and $u_i^R \cdot C(\mathcal{I}_1^2) \neq 0$ for any $i = 1, \dots, n$. Thus, by the PBH eigenvector criteria for controllability and observability, the system (A, B, C) is controllable and observable.

5.3 | Approach (*ii*)

Using Algorithm 1, we have that

$$\begin{array}{r} \mathcal{S}_1 = \{1,4,6,7\} \leftarrow \\ \mathcal{S}_2 = \{3,6,7,8,9,10\} \leftarrow \\ \mathcal{S}_3 = \{3,4,5,6,9,10\} \leftarrow \\ \mathcal{S}_4 = \{1,2,3,5,6,9,10\} \leftarrow \\ \mathcal{S}_5 = \{1,2,3,4,6,7,9,10\} \leftarrow \end{array} \left[\underbrace{\begin{array}{ccccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{array}}_{V^L} \middle| \underbrace{\begin{array}{ccccc} -2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 1 & 1 \end{array}}_{V^R} \right].$$

Thus, the resulting universe is $\mathcal{U} = \bigcup_{i=1}^5 \mathcal{S}_i = \{1, \dots, 10\}$. By solving the associated minimum set covering problem, we obtain as solution $\mathcal{I}_1 = \{1,2,4\}$ (or $\mathcal{I}_2 = \{2,4,5\}$). This solution translates to assign an input to each variable in $\mathcal{I}_1^1 = \{1,2,4\}$ (or $\mathcal{I}_2^1 = \{2,4,5\}$) and to assign an output variable to each variable in $\mathcal{I}_1^2 = \{1,2,4\}$ (or $\mathcal{I}_2^2 = \{2,4,5\}$). This scenario results in input matrices and output matrices such that $\|B\|_0 + \|C\|_0 - \|B \odot C\|_0 = 3 + 3 - 3 = 3$. For instance, for the solution $\mathcal{I}_1$, we obtain the following

$$B(\mathcal{I}_1) = C(\mathcal{I}_1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Remark 1. We draw the attention of the reader to the fact that, in approach (*i*), we had to enumerate all solutions $\mathcal{I}_1^1, \dots, \mathcal{I}_4^1$ using a polynomial approximation algorithm (similar for the observability). Then, one computes the cost for each combination of solutions, leading to a computationally expensive algorithm. Although both approaches found solutions to the MCOP, the approach (*ii*) only requires solving one minimum set covering problem. In other words, the approach we propose in this paper (approach (*ii*)) is much less computationally demanding than the naïve approach (approach (*i*)).

Next, we explore the previous example in the robust scenario stated in Problem 2. In the scenario where one input+output may fail, $s = 1$, we solve the associated Problem 2 using Algorithm 1. We obtain the universe $\mathcal{U} = \{1, \dots, 10\}$, demand function $d(i) = 2$ for $i \in \mathcal{U}$ and the following sets $\mathcal{S}_1 = \mathcal{S}_6 = \{1,4,6,7\}$, $\mathcal{S}_2 = \mathcal{S}_7 = \{3,6,7,8,9,10\}$, $\mathcal{S}_3 = \mathcal{S}_8 = \{3,4,5,6,9,10\}$, $\mathcal{S}_4 = \mathcal{S}_9 = \{1,2,3,5,6,9,10\}$ and $\mathcal{S}_5 = \mathcal{S}_{10} = \{1,2,3,4,6,7,9,10\}$. By solving the associated set multicovering problem, we obtain as solution $\mathcal{I} = \{2,3,4,5,7\}$, which translates into assigning one input and one output to each state variable in $\{2,3,4,5,2\}$. Notice that the obtained solution cannot be achieved by picking two solutions for Problem 1, that is, in general, a minimal rMCOP solution cannot be achieved by "stacking" s solutions to the MCOP. In this example, using two sets from the MCOP would require a union of state variables to observe and control with cardinality equal to 6, whereas the solution to the rMCOP achieves the same with five variables, and assigning inputs to these solutions. By doing so, we would need to control and observe six state variables, instead of five, and the solution would not be minimal.

5.4 | Real-world example

Consider the continuous-time consensus algorithm given as:

$$\dot{x}(t) = -Lx(t), \tag{5}$$

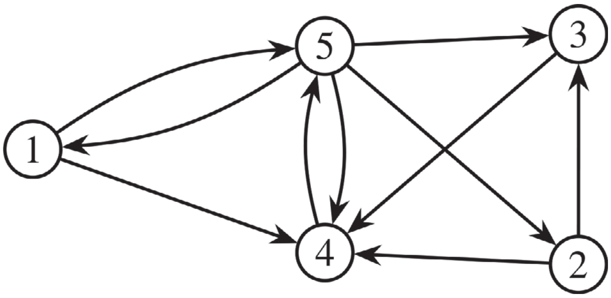

FIGURE 2 Consensus network with five agents

where $x(t) = (x_1(t), \ldots, x_n(t))$, and $x_1(0), \ldots, x_n(0) \in \mathbb{R}$. Moreover, L is the *Laplacian matrix*, $L \in \mathbb{R}^{n\times n}$, where $L_{i,j} = -1$ if $i \neq j$ and there is an edge starting in vertex j and ending in vertex i and $L_{i,i} = d_i$, with d_i the out-degree of vertex i (i.e., the number of edges that start in vertex i). Additionally, consider the network with five agents depicted in Figure 2 with dynamics described by (5).

In the newtwork depicted in Figure 2, the digraph Laplacian matrix is:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & -1 & 2 & 0 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

The eigenvalues of $A = -L$ are $\sigma = \{-4, -3, -2, -1, 0\}$ and hence the matrix has simple eigenvalues. The left-eigenvectors of A are

$$V^L = \begin{bmatrix} | & | & | & | & | \\ v_1^L & v_2^L & v_3^L & v_4^L & v_5^L \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} -2 & -1 & 0 & -1 & 10 \\ -1 & 0 & -1 & 1 & 3 \\ -3 & -1 & 1 & 0 & 1 \\ 6 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 8 \end{bmatrix},$$

and, the right-eigenvectors of A are

$$V^R = \begin{bmatrix} | & | & | & | & | \\ u_1^R & u_2^R & u_3^R & u_4^R & u_5^R \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 & -1 & 1 \\ -1 & -1 & -1 & 2 & 1 \\ -1 & -1 & 3 & 2 & 1 \\ -5 & -1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 & 1 \end{bmatrix}.$$

Consider that we want solve the associated rMCOP problem with $s = 1$. Using Algorithm 1, we have that

$$\begin{array}{r} \mathcal{S}_1 = \{1,2,4,5,6,7,8,9,10\} \leftarrow \\ \mathcal{S}_2 = \{1,3,4,5,6,7,8,9,10\} \leftarrow \\ \mathcal{S}_3 = \{1,2,3,5,6,7,8,9,10\} \leftarrow \\ \mathcal{S}_4 = \{1,2,5,6,7,8,9,10\} \leftarrow \\ \mathcal{S}_5 = \{2,5,6,7,8,10\} \leftarrow \end{array} \left[\begin{array}{ccccc|ccccc} -2 & -1 & 0 & -1 & 10 & -1 & -1 & -1 & -1 & 1 \\ -1 & 0 & -1 & 1 & 3 & -1 & -1 & -1 & 2 & 1 \\ -3 & -1 & 1 & 0 & 1 & -1 & -1 & 3 & 2 & 1 \\ 6 & 1 & 0 & 0 & 2 & -5 & -1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 8 & 3 & 2 & 1 & 0 & 1 \end{array}\right]$$

$$\underbrace{\qquad\qquad}_{V^L} \quad \underbrace{\qquad\qquad}_{V^R}$$

the universe of the minimum set multicovering problem is $\mathcal{U} = \{1, \dots, 10\}$ and $d(i) = s + 1 = 2$ for each $i \in \mathcal{U}$. By solving this minimum set multicovering problem, we obtain as a possible solution $\mathcal{I} = \{1, 2, 3\}$. Thus, one could use numerical matrices:

$$B(\mathcal{I}) = C(\mathcal{I}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Observe that any two nonzero columns of B (or of C) are not orthogonal to the left-eigenvectors (or right-eigenvectors) of A. Hence, by the PBH eigenvectors test for controllability and observability, the system is controllable and observable even when one of the inputs or outputs fail. Further, notice that if $s = 0$, a solution would have to consider two of the sets (e.g., S_1 and S_2). Nonetheless, considering two solutions for the scenario with $s = 0$ would yield four nonzero entries in B and C. Therefore, this solution would not be optimal in the number of actuated and measured state variables.

6 | CONCLUSIONS

In this paper, we have addressed the rMCOP, given the motivation of selecting how many leaders are needed in a MAS for economic reasons. The problem consists of identifying a small number of state variables to be actuated and observed that ensures the system to be both controllable and observable when a specified maximum number of inputs and outputs may fail over time. The naïve solution using the tools in the literature would be to decouple the controllability and observability parts, resulting in a combinatorial solution. However, through an integrated analysis of the two components, we can reduce the task to the computation of a solution to the minimum set multicovering problem. Consequently, we may either explicitly solve a set multicovering problem and obtain the optimal solution to the rMCOP (combinatorial), or approximate the solution resorting to efficient algorithms with close-to-optimality guarantees. We envision as future research considering the same problem for networks evolving in discrete time. Other directions of interest include exploring the case of nonsimple dynamics matrices or adding a cost function to account for the economics or restrictions associated with the mission to be carried by the MAS.

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DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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