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Self-triggered state-feedback control of linear plants under bounded disturbances

J. Almeida[1,*,†], C. Silvestre[1,2] and A. M. Pascoal[1]

[1] *Institute for Robotics and Systems in Engineering and Science (LARSyS), Instituto Superior Técnico, Universidade de Lisboa, Lisboa, Portugal*
[2] *Department of Electrical and Computer Engineering, Faculty of Science and Technology of the University of Macau, Macau, China*

SUMMARY

This paper addresses the problem of self-triggered state-feedback control for linear plants under bounded disturbances. In a self-triggered scenario, the controller is allowed to choose when the next sampling time should occur and does so based on the current sampled state and on a priori knowledge about the plant. Besides comparing some existing approaches to self-triggered control available in the literature, we propose a new self-triggered control strategy that allows for the consideration of model-based controllers, a class of controllers that includes as a special case static controllers with a zero-order hold of the last state measurement. We show that our proposed control strategy renders the solutions of the closed-loop system globally uniformly ultimately bounded. We further show that there exists a minimum time interval between sampling times and provide a method for computing a lower bound for it. An illustrative example with numerical results is included in order to compare the existing strategies and the proposed one.

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1. INTRODUCTION

With the advent of digital devices with considerable computational capabilities, controller implementation has shifted from continuous time to discretized sampled-data strategies. For this reason, special techniques have been developed for the analysis and synthesis of control systems whose measurements are not available continuously. The particular case where measurements are available periodically (periodic sampling) has been studied extensively in the literature. For example, [1, 2] cover exhaustively exact discretization and emulation designs for linear systems while [3, 4] focus on emulation designs for nonlinear systems. Nonetheless, for many systems of interest periodic sampling is not the proper strategy to adopt. Such is the case with networked control systems, where the lengths of the sampling intervals are often dependent on exogenous signals and can therefore be neither equal nor defined in advance. In this case, a controller may be allowed to choose the next sampling time (also known as update or release time), which effectively plays the role of an extra DOF in the controller design process. Control techniques of this kind fall roughly into the two categories shown in Figure 1: event-triggered and self-triggered control.

In the first case, an event detector is responsible for testing if a triggering condition (basically, a function of the plant's state) is true or false. If true, then a sampling event is triggered. The advantage of this approach versus a periodic sampling strategy is that the control input is only modified when

*Correspondence to: J. Almeida, ISR, Instituto Superior Técnico, Av. Rovisco Pais, 1, 1049-001 Lisboa, Portugal.
†E-mail: jalmeida@isr.ist.utl.pt

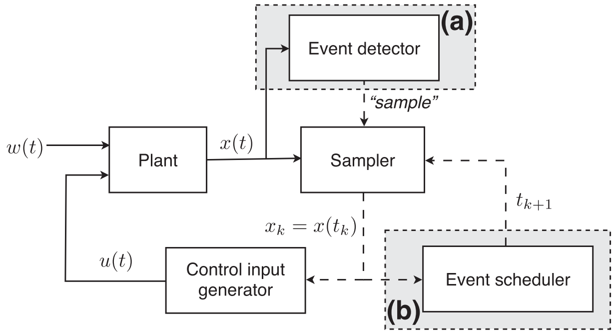

Figure 1. Triggered control systems: event-triggered (block **(a)** active, a sampling event is triggered when the sampler receives a 'sample' message); self-triggered (block **(b)** active, a sampling event will be triggered when $t = t_{k+1}$). The state and the control input of the plant and the exogenous state disturbances are represented by x, u, and w, respectively. Solid lines denote continuous-time signals, while dashed lines denote signals that are only updated at sampling times.

some relevant change of the plant's state occurs and this typically leads to a reduction of the number of samples required to achieve desired specifications. Literature on event-triggered control is quite extensive at this point. The interested reader is referred to [5–12] and references therein. The current state of the art in event-triggered control can be illustrated by [9–11]. The solutions proposed in these three references address output-feedback problems in the presence of disturbances and introduce event mechanisms not only between the sensor and the controller, but also between the controller and the actuator. In [9] an impulsive system formulation is proposed to address the problem of output-feedback control of continuous-time plants with guaranteed performance as measured by the $\mathcal{L}_\infty$-gain of the closed-loop system. In [10], the authors use a different framework, choosing to formulate the same problem using delayed systems theory and using as performance index the $\mathcal{L}_2$-gain. Finally, in [11] reference tracking control problems are addressed for discrete-time systems where state-output stability of the error system and internal stability of the feedback system are guaranteed. We should point out that, at present, there do not exist self-triggered versions of the results reported in [9–11].

In order to avoid monitoring the state or the output constantly, as required in event-triggered control, self-triggered control strategies have been proposed [13–18]. The idea put forth in [13] exploits the fact that instead of continuously testing a triggering condition, an event scheduler (Figure 1) may compute when the next sampling event should occur, based on the current sampled state and knowledge about the plant dynamics.

This paper addresses the problem of self-triggered state-feedback control for linear plants under bounded disturbances as represented by the block diagram of Figure 1. We discuss existing self-triggered approaches and propose a new self-triggered control strategy. In an attempt to reduce the number of updates required to keep the closed-loop system at a desired performance level, we consider model-based controllers (see, e.g., [19]), which include as a special case static controllers with a zero-order hold (ZOH) of the last state measurement. The idea of using model-based controllers to improve performance has also been applied in [8] in the context of event-triggered control. The propose self-triggered control strategy is inspired by the work in [15]. The motivation to derive a self-triggered strategy different from the one proposed in [15] stems from the fact that for a particular choice of model-based controller, the approach taken in [15] degenerates into periodic sampling. Although the authors in [15] obtain stronger stability properties for the closed-loop system, simulation results presented in Section 4 demonstrate that a clear gain in performance is observed when using our proposed control strategy. We show that our proposed control strategy renders the solutions of the closed-loop system globally uniformly ultimately bounded (GUUB) (defined later in the text). We further show that there exists a minimum time interval between sampling times and provide a method for computing a lower bound for it.

The paper is organized as follows. In Section 2 we describe the type of plants under consideration, discuss the differences between event-triggered and self-triggered control, and formally state the control problem addressed. In Section 3, we present some details on existing self-triggered control strategies and discuss their differences. We then proceed to describe our proposed self-triggering control strategy, along with its stability analysis. In Section 4, an illustrative example is used to compared existing approaches and our proposed approach through simulation results. Finally, Section 5 contains concluding remarks. For clarity of presentation, all proofs have been placed in Appendix A.

2. CONTROL PROBLEM FORMULATION

Consider a linear time-invariant plant with state $x \in \mathbb{R}^n$ and initial state $x(t_0) = x_0$ that satisfies, for all $t \geqslant t_0$,

$$\dot{x}(t) = Ax(t) + B_1 u(t) + B_2 w(t), \tag{1}$$

where $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^q$ is an exogenous disturbance, and A, B_1, and B_2 are matrices of appropriate dimensions. With loss of generality, we take $t_0 = 0$. In (1), we assume that the disturbance is bounded, that is, $\|w\|_{\mathcal{L}_\infty} < \infty$ where $\|x\|_{\mathcal{L}_\infty}$ is the $\mathcal{L}_\infty$ norm of a signal $x(t)$, defined as $\sup_{t \geqslant 0} \|x(t)\|$ with $\|\cdot\|$ denoting the Euclidean norm. The pair (A, B_1) is assumed to be controllable.

In this paper, we focus on sampled-data control strategies that can be described with the help of the block diagram of Figure 1. In this setup, the plant's state x is sampled whenever $t = t_k$, where $\{t_k\}_{k \geqslant 1}$ denotes a sequence of sampling times. This information is then sent to the control input generator and the event scheduler. Given the last sampled state $x_k = x(t_k)$, the control input generator computes which signal $u(t)$ should be applied to the plant between sampling times. It is designed such that desired stability or performance specifications are satisfied when a continuous-time controller with access to full-state measurements is employed. Based on the last sampled state and on knowledge about the plant dynamics, the event scheduler computes when the next sampling time t_{k+1} should occur and communicates this information back to the sampler. The role of the event scheduler is to guarantee that the stability or performance properties that hold when a continuous controller is applied are recovered (in a well-defined sense) when a self-triggered controller is used instead.

In what follows, we present in greater detail the structure of the control input generator and the differences between event-triggered and self-triggered control, before formally stating the control problem addressed in this paper.

2.1. *Control input generator*

Let the gain matrix K be such that $A + B_1 K$ is Hurwitz (this is always possible because the pair (A, B_1) is assumed to be controllable). The control input generator computes $u(t)$, for all $k \geqslant 0$ and all $t \in [t_k, t_{k+1})$, according to

$$t \in [t_k, t_{k+1}) : \dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}_1 u(t) \tag{2a}$$

$$u(t) = K\hat{x}(t) \tag{2b}$$

$$t = t_k : \hat{x}(t_k) = x(t_k), \tag{2c}$$

where $\hat{x} \in \mathbb{R}^n$ is an internal state variable. A control input that is kept constant between sampling times is obtained by making $\hat{A}$ and $\hat{B}_1$ equal to zero. We will refer to this way of generating u as a ZOH. We would like to point out that holding the value of u constant between sampling times, as in [15], is just one possibility. Another alternative is to include a model of the plant in the controller as in model-based control [19]. The model simulates the plant state evolution in between sampling times when no information about the plant's actual state is available. This occurs when we take $\hat{A} = A$ and $\hat{B}_1 = B_1$, obtaining what we shall refer to as the *exact model hold* (EMH). Even though the EMH controller may consume more energy than the ZOH one, it should lead to better

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closed-loop performance. Assuming perfect knowledge of A and B_1 does not impose any added restriction because, to compute the next sampling time, we need to know exactly A and B_1. Nonetheless, in general, we will not have the exact knowledge of A and B_1, which raises questions about robustness to parameter uncertainty. Such issues are not addressed here but provide fertile ground for future research.

2.2. *Event-triggered versus self-triggered*

In event-triggered control, the next sampling time is defined implicitly as the time when some significant event occurs. For example, consider a continuous event function $g : \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that if, for all $k \geqslant 0$ and all $t \in [t_k, t_{k+1})$, $g(t - t_k, x(t), \hat{x}) \leqslant 0$, then the closed-loop system behaves as desired. Therefore, whenever the function g reaches zero, a new sampling action is triggered, and the control input is updated. That is, at a given sampling time t_k, after measuring state x_k, the next sampling time is implicitly defined by

$$t_{k+1} = \min\{t > t_k : g(t - t_k, x(t), \hat{x}(t)) = 0\}. \tag{3}$$

As mentioned in Section 1, event-triggered control requires one to constantly monitor the plant's state for changes as demonstrated by the dependence of the function g on $x(t)$. To avoid this, self-triggered control strategies have been proposed where the event scheduler is replaced by an event detector that determines when the next sampling time should occur and communicates this information to the sampler. To go from one control strategy to the other, the idea is to construct an event scheduler out of an existing event detector. Because access to $x(t)$ is barred, the function g in (3) must be replaced by a continuous function $h : \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that, for all $t \geqslant t_k$,

$$h(t - t_k, x_k, \hat{x}(t)) \leqslant 0 \Rightarrow g(t - t_k, x(t), \hat{x}(t)) \leqslant 0. \tag{4}$$

Ideally, we would like to have $g(t - t_k, x(t), \hat{x}(t)) = h(t - t_k, x_k, \hat{x}(t))$ for all $t \geqslant t_k$. However, this is impossible because $x(t)$ depends not only on x_k but also on the disturbance $w(t)$ that is assumed unknown and cannot be measured. For each function h, we would like to arrive at an explicit formula for the next sampling time t_{k+1} as a function of the current measured state. This would yield a self-triggered sampling strategy where, at time $t = t_k$, the event scheduler determines when the next sampling time t_{k+1} should occur based on the current state and on knowledge about the plant dynamics. Although this is sometimes possible, in general, such explicit formula cannot be obtained in closed form because of the transcendental dependence of $x(t)$ on time through a matrix exponential. Nonetheless, as we shall see, it is possible to solve (3) with g replaced by h approximately while still guaranteeing desired stability and performance requirements. The computations carried out by the scheduler to find this approximate solution are represented by a *scheduling function* $\tau : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ that maps states to time intervals such that, for all $k \geqslant 0$,

$$\tau_k = t_{k+1} - t_k = \tau(x_k). \tag{5}$$

2.3. *Problem formulation*

To formally state the problem addressed, we borrow the following notion of stability from [20, Chapter 4].

Definition (GUUB)
The solutions of (1) (with $u = 0$) are GUUB with ultimate bound b if there exists a positive constant b, independent of $t_0 \geqslant 0$, and for every $a > 0$, there is $T = T(a, b) \geqslant 0$, independent of t_0, such that

$$\|x(t_0)\| \leqslant a \Rightarrow \|x(t)\| \leqslant b, \quad \forall\, t \geqslant t_0 + T. \tag{6}$$

Problem 1
Consider a linear plant with bounded disturbances as in (1) and a control input generator as in (2). Find a scheduling function τ such that, when the sequence of sampling times $\{t_k\}_{k \geqslant 1}$ is generated according to (5), the solutions of the closed-loop system formed by (1), (2), and (5) are GUUB.

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Ideally, the problem formulation should also aim at minimizing the number of samples taken over some time interval. This is a harder problem that is not addressed here but deserves further research. In the following section, we present a few self-triggered controllers that provide a solution to Problem 1.

3. CONTROL STRATEGIES

In this section, we present some existing self-triggered control strategies that solve Problem 1 and propose a new self-triggered control strategy that avoids some shortcomings of the existing ones.

3.1. *Periodic control*

The classical control strategy is to consider periodic sampling, which is obtained by taking $\tau(x) = T_s$ for all $x \in \mathbb{R}^n$, where $T_s > 0$ is a fixed sampling period. In this case, it can be shown that the solutions of the closed-loop system are GUUB as long as the closed-loop system, in the absence of disturbances, is asymptotically stable. Both types of holds can be utilized but, in the absence of disturbances, while periodic control with a ZOH is unstable for sampling periods greater than a certain critical sampling period, with an EMH, the closed-loop system is always asymptotically stable.

3.2. *Self-triggered strategy in* [16, 17]

In [16, 17], the authors propose a self-triggered control strategy that renders the closed-loop system finite-gain $\mathcal{L}_2$ stable from disturbance to state for any bounded disturbance in $\mathcal{L}_2$ space. The control input is applied in a ZOH manner ($\hat{A}$ and $\hat{B}_1$ equal to zero in (2a)), and the scheduling function $\tau : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ is of the form

$$\tau(x) = \frac{1}{\mu} \ln\left(1 + \frac{\mu(x^\top N_2 x)^{\frac{1}{2}}}{(x^\top A_{cl}^\top N_1 A_{cl} x)^{\frac{1}{2}}}\right), \tag{7}$$

where $A_{cl} = A + B_1 K$, N_1 and N_2 are some positive definite matrices, and μ is a positive scalar. Note that there exist positive scalars $\tau_{\min}$ and $\tau_{\max}$ such that $0 < \tau_{\min} \leqslant \tau(x) \leqslant \tau_{\max}$ for all $x \in \mathbb{R}^n$.

3.3. *Self-triggered strategy in* [15]

Because our proposed self-triggered control strategy is closely related with the one reported in [15], we describe it here in greater detail. We then proceed to describe our self-triggered strategy and establish its stability properties.

The self-triggered strategy present in [15] works by first deriving a control strategy in the absence of disturbances, and then studying *a posteriori* the effect of disturbances on the closed-loop system. Hence, assume for the moment that $w \equiv 0$. For a given positive definite matrix Q, let P be the positive definite solution to the Lyapunov equation

$$(A + B_1 K)^\top P + P(A + B_1 K) + Q = 0. \tag{8}$$

Such a P always exists because $A + B_1 K$ is Hurwitz. Consider the function $V : \mathbb{R}^n \to \mathbb{R}$ defined as

$$V(x) = x^\top P x, \tag{9}$$

where P is given by (8). Let $\lambda_c = \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}$. Under continuous feedback of the form $u(t) = Kx(t)$, we have that, for all $x_0 \in \mathbb{R}^n$ and all $t \geqslant t_0$,

$$V(x(t)) \leqslant V(x_0) e^{-\lambda_c (t - t_0)}, \tag{10}$$

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that is, the closed-loop system with continuous state feedback is globally uniformly exponentially stable (GUES). We shall refer to λ_c as the (continuous time) decay rate of V.

Consider now the case where the control input is kept constant between sampling times, that is,

$$u(t) = Kx_k, \tag{11}$$

for all $t \in [t_k, t_{k+1})$ and all $k \geqslant 0$. Let $\lambda_\alpha = (1-\alpha)\lambda_c$ be a desired decay rate for V where $\alpha \in (0,1)$. Note that $0 < \lambda_\alpha < \lambda_c$. If the sequence of sampling times $\{t_k\}_{k\geq 1}$ is such that

$$V(x(t)) \leqslant V(x_k)e^{-\lambda_\alpha(t-t_k)}, \tag{12}$$

holds for all $t \in [t_k, t_{k+1})$ and all $k \geqslant 0$, then the function V will satisfy

$$V(x(t)) \leqslant V(x_0)e^{-\lambda_\alpha(t-t_0)}, \tag{13}$$

for all $t \geqslant t_0$. To compute t_{k+1} such that (12) holds for all $t \in [t_k, t_{k+1})$, the event scheduler simulates the evolution of $x(t)$ by using a copy of the plant's dynamics (1) with the control input as in (11). Given $x \in \mathbb{R}^n$, let $\xi_x \in \mathbb{R}^n$ satisfy, for all $\delta \geqslant 0$,

$$\frac{\mathrm{d}}{\mathrm{d}\delta}\xi_x(\delta) = A\xi_x(\delta) + B_1 Kx, \quad \xi_x(0) = x. \tag{14}$$

Consider the function $h_{VS} : \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n \to \mathbb{R}$ defined as

$$h_{VS}(\delta, x) = V(\xi_x(\delta)) - S(\delta, V(x)), \tag{15}$$

where the function $S : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ is defined as

$$S(\delta, v) = ve^{-\lambda_\alpha \delta}. \tag{16}$$

Let the function $\tau : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ be defined as

$$\tau(x) = \max\{0 \leqslant \tau_2 \leqslant \tau_{\max} : h(\tau_1, x) \leqslant 0, \text{for all } 0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \tau_{\max}\}. \tag{17}$$

Then, the next sampling time is computed according to (5) where the scheduling function is τ_{VS} defined as (17) with $h = h_{VS}$. Here, $\tau_{\max}$ is a design parameter. The function τ_{VS} is such that the interval $[t_k, t_{k+1})$ where (12) holds is of maximal length. It is shown in [15] that consecutive sampling times are separated by a minimum time interval, that is, for all $x \in \mathbb{R}^n$, there exists a positive constant $\tau^*_{\min}$ such that $\tau_{VS}(x) \geqslant \tau^*_{\min}$. An implicit formula for the computation of $\tau^*_{\min}$ is given in [15, Lemma 4.1], where it is also shown that if the sampling intervals are scheduled using τ_{VS}, then the closed-loop system is GUES. When disturbances are present, the sampling times are still scheduled using τ_{VS} as if no disturbances were present. It is shown in [15] that in this case, the closed-loop system is input-to-state stable with respect to the disturbance w.

Recall the control input generator described in Section 2.1. Note that in the absence of disturbances, after a single sampling instant occurs, the state $\hat{x}$ of the EMH will be equal to the actual state x of the plant for all time. Because the control strategy reported in [15] schedules sampling events as if no disturbances were present, with an EMH, a sampling event is always scheduled with $\tau_k = \tau_{\max}$ and thus degenerates into periodic sampling with a fixed period of $\tau_{\max}$. One advantage of the control strategy proposed in Section 3.5 is that controllers with self-triggered characteristics are obtained for both ZOH and EMH.

3.4. Self-triggered strategy in [18]

In [18], the authors address the state-feedback control of a linear plant in the absence of disturbances where the aim is to enlarge as much as possible the sampling intervals. They begin by partitioning the state space into a set of conic regions $\mathcal{R}_s = \{x \in \mathbb{R}^n : x^\top Q_s x \geqslant 0\}$, where Q_s is a symmetric matrix and $s \in \{1, \dots, q\}$. To each region, a constant sampling interval is associated. The scheduling function is then defined as equal to the sampling interval corresponding to the conic region where the current sampled state is located, that is, $\tau(x) = \tau_s$ for all $x \in \mathcal{R}_s$ and $s \in \{1, \dots, q\}$. Some extra computations are thus required by this strategy to determine to which region the current sampled state belongs to. The authors attempt to enlarge the average sampling interval by increasing as much as possible the sampling intervals τ_s in each conic region $\mathcal{R}_s$. It is shown that the closed-loop system is rendered exponentially stable with solutions tending to zero with a chosen rate of decay.

In [18], the control input is applied in a ZOH manner. If the ZOH is replaced by an EMH, then the strategy reported in [18] will suffer from the same issue pointed out for the strategy presented in [15]. In the absence of disturbance, if an EMH is employed, then the closed-loop system is guaranteed to decay at a desired rate. As such, no need for sampling is required to fulfill the rate decay requirement, and the maximal sampling interval in each region will be equal to the design parameter $\tau_{\max}$, resulting in periodic sampling at the maximal sampling interval allowed.

3.5. Proposed self-triggered control strategy

In the presence of disturbances and under continuous feedback of the form $u(t) = Kx(t)$, it is possible to show (see, e.g., [20, Lemma 9.2]) that the function V defined in (9) satisfies

$$V(x(t)) \leqslant \begin{cases} V(x_0)e^{-\lambda_\vartheta (t-t_0)}, & \text{if } t_0 \leqslant t < t_0 + T \\ W_\theta, & \text{if } t \geqslant t_0 + T \end{cases}, \tag{18}$$

for some finite $T = T(V(x_0), W_\theta, \lambda_\vartheta) \geqslant 0$, where $0 < \theta < \vartheta < 1$

$$W_\theta = \lambda_{\max}(P) \left(\frac{2\varpi}{\theta \lambda_c} \right)^2, \tag{19}$$

and $\varpi = \|B_2 w\|_{\mathcal{L}_\infty}$. Note that (18) is equivalent to

$$V(x(t)) \leqslant \max\{V(x_0)e^{-\lambda_\vartheta (t-t_0)}, W_\theta\}, \forall t \geqslant t_0. \tag{20}$$

Instead of ignoring the disturbances' effect on the plant's state evolution during the design of the scheduling function, we take into account its effect by performing three changes to the scheduling scheme presented in the preceding section. First, we replace the function S defined in (16) by one that resembles the expected continuous-time performance with disturbances. For this reason, consider the function $R : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ defined as

$$R(\delta, v) = \max\{v e^{-\lambda_\vartheta \delta}, W_\theta\}. \tag{21}$$

Second, to handle other hold devices as the ones described in Section 2.1, we redefine (14) as

$$\frac{\mathrm{d}}{\mathrm{d}\delta} \xi_x(\delta) = A\xi_x(\delta) + B_1 K e^{(\hat{A} + \hat{B}_1 K)\delta} x, \quad \xi_x(0) = x. \tag{22}$$

Finally, we replace the function h_{VS} defined in (15) by the function $h_{UR} : \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n \to \mathbb{R}$ defined as

$$h_{UR}(\delta, x) = U(\delta, \xi_x(\delta)) - R(\delta, V(x)), \tag{23}$$

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where the function $U : \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n \to \mathbb{R}$ is such that $V(x(t_k + \delta)) \leqslant U(\delta, \xi_{x_k}(\delta))$ for all $\delta \geqslant 0$. To find such U, note that the time response of (1) for $t \in [t_k, t_{k+1})$ can be written as

$$x(t_k + \delta) = \xi_{x(t_k)}(\delta) + \zeta(t_k, \delta), \tag{24}$$

where $\xi_{x(t_k)}(\delta)$ denotes the time response of the closed-loop system in the absence of disturbances and $\zeta(t_k, \delta)$ accounts for the disturbances' effect on the state evolution. It is easy to see that, for all $\delta \geqslant 0$,

$$\zeta(t, \delta) = \int_t^{t+\delta} e^{A(t+\delta-\eta)} B_2 w(\eta) \mathrm{d}\eta = \int_0^{\delta} e^{A(\delta-\nu)} B_2 w(t+\nu) \mathrm{d}\nu. \tag{25}$$

The norm of $\zeta(t, \delta)$ satisfies

$$\|\zeta(t,\delta)\| \leqslant \varpi \int_0^{\delta} \|e^{A(\delta-\nu)}\| \mathrm{d}\nu \leqslant \varpi \int_0^{\delta} e^{\varsigma(\delta-\nu)} \mathrm{d}\nu = \begin{cases} \frac{\varpi}{\varsigma}\left(e^{\varsigma\delta} - 1\right), & \varsigma > 0 \\ \varpi\delta, & \varsigma = 0 \end{cases} =: \beta(\delta) \tag{26}$$

where $\varsigma \in \mathbb{R}$ satisfies $\|e^{At}\| \leqslant e^{\varsigma t}$ for all $t \geqslant 0$. Evaluating (9) with x given by (24) and using (26) yields

$$\begin{aligned} V(x(t_k + \delta)) &= V(\xi_{x_k}(s)) + \zeta^{\top}(t_k, \delta) P(2\xi_{x_k}(\delta) + \zeta(t_k, \delta)) \\ &\leqslant V(\xi_{x_k}(\delta)) + \beta(\delta)(2\|P\xi_{x_k}(\delta)\| + \lambda_{\max}(P)\beta(\delta)) \\ &=: U(\delta, \xi_{x_k}(\delta)), \end{aligned} \tag{27}$$

where the bounding function $U(\delta, x)$ is of the form $V(x) + M(\delta, x)$ with

$$M(\delta, x) = \beta(\delta)(2\|Px\| + \lambda_{\max}(P)\beta(\delta)). \tag{28}$$

Finally, the scheduling function becomes τ_{UR} that is defined as (17) with $h = h_{UR}$.

Note that in the absence of disturbances ($\varpi = 0$), we have $W_\theta = 0$ and $M \equiv 0$. Therefore, $S(\delta, v) = R(\delta, v)$ and $U(\delta, x) = V(x)$, meaning that the sampling strategy presented in [15] and the one proposed here generates the same sequence of sampling times.

The following theorem shows that the proposed scheduling strategy does indeed solve Problem 1.

Theorem 1 (Solutions of the closed-loop system are GUUB)
Consider the closed-loop system given by (1) and (2). If the sequence of sampling times $\{t_k\}_{k \geqslant 1}$ is generated using τ_{UR}, then the solutions of the closed-loop system are GUUB with ultimate bound $b = \sqrt{\kappa(P)\frac{2\varpi}{\theta\lambda_c}}$. Moreover, the sequence of sampling intervals $\{\tau_k\}_{k=0}^{+\infty}$ is bounded in the succeeding texts by some $\tau^*_{\min} > 0$.

Although the previous theorem guarantees the existence of a minimum sampling interval $\tau^*_{\min}$, from an implementation point of view, it is of crucial importance to compute the value of $\tau^*_{\min}$ or at least a lower bound for it.

Lemma 1 (Lower bound on the minimum sampling interval)
Suppose the hypothesis of Theorem 1 holds and that $x_0 \in \Omega_c = \{x \in \mathbb{R}^n : V(x) \leqslant c\}$ for some $c > 0$. Then the sequence of sampling times $\{t_k\}_{k \geqslant 1}$ is such that $t_{k+1} - t_k \geqslant \tau^*_{\min} \geqslant \tau_{\min}$ for all $k \geqslant 0$, where $\tau_{\min}$ is implicitly defined as

$$\tau_{\min} = \min\{\delta > 0 : H(\delta, v) = 0, 0 \leqslant v \leqslant c\}. \tag{29}$$

The function $H : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ is defined as

$$H(\delta, v) = G(\delta, v) + \bar{M}(\delta, v) - R(\delta, v), \tag{30}$$

where the function R is defined in (21) and the functions $G, \bar{M}, E, \Xi : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ are defined as

$$G(\delta, v) = ve^{-\lambda_c\delta} + \frac{1}{\lambda_c}(1 - e^{-\lambda_c\delta})2\Xi(\delta, v)\|B_1 K\| E(\delta, v) \tag{31}$$

$$\bar{M}(\delta, v) = \beta(\delta)\left(2\Xi(\delta, v) + \lambda_{\max}(P)\beta(\delta)\right) \tag{32}$$

$$E(\delta, v) = \frac{\|\tilde{A} + \tilde{B}_1 K\|}{\gamma_1 - \gamma_2}\left(e^{\gamma_1 \delta} - e^{\gamma_2 \delta}\right)\sqrt{\frac{v}{\lambda_{\min}(P)}} \tag{33}$$

$$\Xi(\delta, v) = e^{\gamma_3 \delta}\sqrt{\lambda_{\max}(P) v} + \frac{\|PB_1 K\|}{\gamma_1 - \gamma_3}\left(e^{\gamma_1 \delta} - e^{\gamma_3 \delta}\right)\sqrt{\frac{v}{\lambda_{\min}(P)}}, \tag{34}$$

with $\tilde{A} = A - \hat{A}$, $\tilde{B}_1 = B_1 - \hat{B}_1$, $\gamma_1 = \mu(\hat{A} + \hat{B}_1 K)$, $\gamma_2 = \mu(A)$, and $\gamma_3 = \mu(PAP^{-1})$.

The value of $\tau_{\min}$ is given as the zero on a nonlinear function with two scalar arguments. Even for a fixed v, there is in general no closed-form expression for the smallest zero of $H(\delta, v)$. However, numerical solvers or root finders can be used to find a good approximation given an interval where a zero is guaranteed to exist. The actual computation of $\tau_{\min}$ is carried out by solving $H(\delta, v) = 0$ for fixed values of v over a grid of values in the interval $[0, c]$ because we assume that $x_0 \in \Omega_c$ for some $c > 0$. When $v > W_\theta$, we further assume that the function R is always a decaying exponential to facilitate the computation of the zero (it is easier when the function is smooth because this avoids the non-differentiability at $v = W_\theta$) that is guaranteed to be a lower bound on the actual value.

To evaluate (17), the solution of (22) has to be computed, and this must be performed whenever a sampling action is carried out. Hence, determining t_{k+1} can become computationally intensive. To mitigate this issue, a gridding approach is employed. The actual implementation of (17) uses a gridded event scheduler that computes the next sampling interval by replacing the function τ in (17) with the function $\tau_{\text{grid}} : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$, defined as

$$\tau_{\text{grid}}(x) = \tau_{\min} + \max\{0 \leqslant j_2 \leqslant J : h(\tau_{\min} + j_1 \Delta, x) \leqslant 0 \text{ for all } 0 \leqslant j_1 \leqslant j_2 \leqslant J\}\Delta \tag{35}$$

where $\Delta > 0$ and $0 < \tau_{\min} \leqslant \tau^*_{\min}$ are design parameters and $J = \lfloor(\tau_{\max} - \tau_{\min})/\Delta\rfloor$.

4. AN ILLUSTRATIVE EXAMPLE

To compare different types of sampling strategies, we shall use a four-dimensional linearized model of an unstable batch reactor process presented in [21]. The plant is modeled as a linear system where

$$A = \begin{bmatrix} 1.38 & -0.2077 & 6.715 & -5.676 \\ -0.5814 & -4.29 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix}, B_1 = \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix}, B_2 = I_4.$$

We consider the following control strategies.

- Periodic control (`periodic-{ZOH, EMH}`) with sampling rate $\frac{1}{T_s}$ in the ranges $[1.809, 3.25]$ and $[0.25, 3.25]$ for controllers using a ZOH and an EMH, respectively.
- Self-triggered control strategy in [15] (`[12]-ZOH`) with a ZOH and adjustable parameter α in the range between 0 and 0.9 with a 0.1 step size. The minimum sampling interval $\tau^*_{\min}$ associated with each value of α is computed as described in [15, Lemma 4.1]. The values obtained are given in Table I, along with the values of the gridding parameters $\tau_{\min}$ and Δ used in the simulations. The maximum sampling interval is set to $\tau_{\max} = 1$.
- Self-triggered control strategies proposed in Section 3.5 (`UR-Self-{ZOH, EMH}`) with adjustable parameters $\vartheta, \theta \in (0, 1)$ (in the simulations, we set $\theta = \frac{\vartheta}{1.1}$) using a ZOH and an EMH. Although the main focus is on the proposed self-triggered strategy, it is also possible to implement an event-triggered control strategy with event function given by $g(\delta, x, y) = V(x) - R(\delta, V(y))$ where V and R are defined in (9) and (21), respectively. We implement it for the EMH case (`UR-Event-EMH`) to discuss how well the self-triggered strategy approximates its event-triggered counterpart. For the self-triggered strategy, when using a ZOH, we choose $\tau_{\min}$ according to Table II, $\Delta = 10^{-2}$, $\tau_{\max} = 1$, and ς in (26) equal to $(\mu(A) + \|A\|)/2$. When using an EMH, we set $\tau_{\min} = 5 \times 10^{-3}$, $\Delta = 5 \times 10^{-2}$, $\tau_{\max} = 5$, and $\varsigma = \mu(A)$.

Table I. Gridding parameters $\tau_{\min}$ and Δ (value $\times\ 10^{-3}$) for self-triggered control strategy in [15] as a function of α.

α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\tau_{\min}$	5	5	10	15	20	20	30	30	30
Δ	5	5	5	5	10	10	10	10	10

Table II. Minimum time $\tau_{\min}$ (value $\times\ 10^{-5}$) for the proposed strategy (using a ZOH) as a function of ϑ.

ϑ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\tau_{\min}$	1	3	4	6	7	9	10	10	10

All strategies use a state-feedback controller designed to place the closed-loop eigenvalues (of closed-loop system with continuous feedback) at $\{-3 \pm i1.2, -3.6, -3.9\}$, yielding the gain matrix

$$K = \begin{bmatrix} 0.1006 & -0.2469 & -0.0952 & -0.2447 \\ 1.4099 & -0.1966 & 0.0139 & 0.0823 \end{bmatrix}.$$

The matrix P in (8) is obtained by setting $Q = I_4$. With continuous feedback, the closed-loop system has a rate of decay $\lambda_c = 0.8253$.

The criterion used to compare the performance of each control strategy is defined as

$$\mathrm{UB} = \sup_{\|w\|_{\mathcal{L}_\infty} \leqslant \varpi} \lim_{t\to+\infty} \|x(t)\|, \tag{36}$$

with $\varpi > 0$. The value of UB is an ultimate bound on the solutions of the closed-loop system after transients have subsided.

A series of simulations were carried out to compare the performance of all sampling schemes. A total of $N_{\text{sims}} = 40$ simulations where each had a total duration of $T_{\text{sim}} = 500$ were carried out for each sampling strategy. The disturbance w is generated by multiplying two signals: a uniformly distributed signal over the interval $[0, \varpi]$ with $\varpi = 10$ and uniformly distributed unit vectors in $\mathbb{R}^4$. The initial state x_0 was uniformly generated on a sphere of radius 10^3. The value of UB in (36) is approximated as

$$\mathrm{UB}^{(\text{sim})} = \max_{i\in\{1,\ldots,N_{\text{sims}}\}} \max_{T_{\text{trans}} \leqslant t \leqslant T_{\text{sim}}} \|x^{[i]}(t)\|, \tag{37}$$

where $x^{[i]}(t)$ is the plant's state response of the ith simulation and T_{trans} is the time interval deemed necessary for transient effects to subside ($T_{\text{trans}} = 200$). The values of $\mathrm{UB}^{(\text{sim})}$ computed for each strategy are plotted in Figure 2 against the average sampling rate observed‡.

For the `periodic-ZOH` strategy, $\frac{1}{T_s} = 1.809$ is very close to instability, so for smaller sampling rates, the closed-loop system will become unstable, and the bound will become infinity. On the other hand, the `periodic-EMH` strategy is asymptotically stable for all sampling rates, but, as expected, the bound grows as the sampling rate decreases. As expected, the value of $\mathrm{UB}^{(\text{sim})}$ for `periodic-{ZOH, EMH}` tends to the continuous bound as the sampling rate increases. The proposed strategies `UR-Self-{ZOH, EMH}` perform similarly to the corresponding `periodic-{ZOH, EMH}` controllers. Nonetheless, the control strategy `UR-Event-EMH` outperforms `periodic-EMH` in a certain range of average sampling rates as allowed by changes in the parameter ϑ. The `UR-Self-EMH` version does not perform as well, which suggests that conservativeness introduced in the derivation of the self-triggered strategy is eliminating some of the

‡We observed in simulation that the control strategy proposed in [16, 17] generated very high average sampling rates when compared to the previous strategies, yielding a value of $\mathrm{UB}^{(\text{sim})}$ very close to the one obtained with a continuous controller. Hence, we decided to focus on a range of sampling rates around the stability limit of periodic control with a ZOH where changes in $\mathrm{UB}^{(\text{sim})}$ are more apparent and therefore do not present results for the control strategy in [16, 17].

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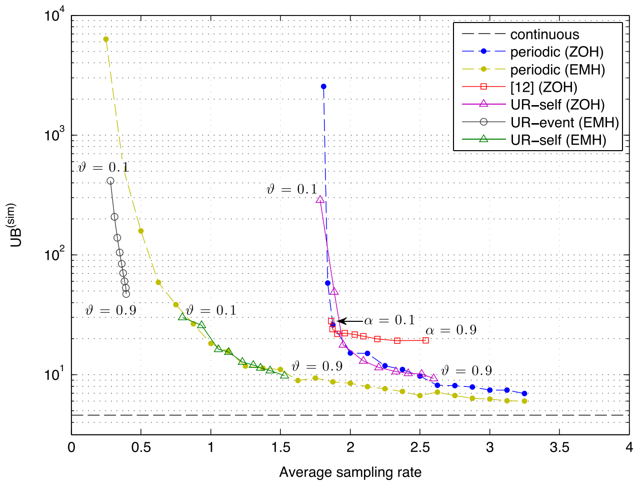

Figure 2. Simulated ultimate bounds $\mathrm{UB}^{(\mathrm{sim})}$ as a function of the average sampling rate for different control strategies (as a reference, $\mathrm{UB}^{(\mathrm{sim})}_{\texttt{continuous}} = 4.5879$).

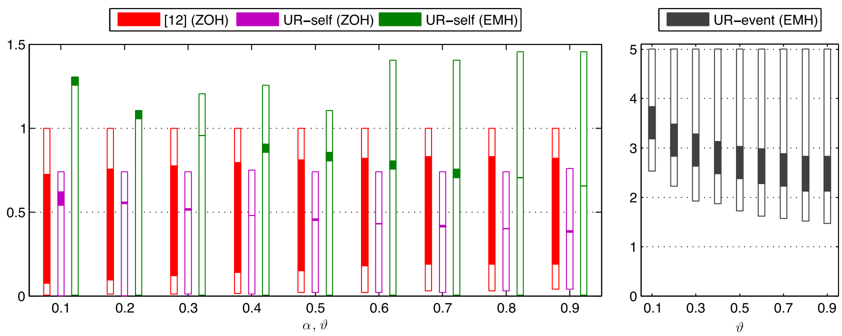

Figure 3. Statistics related with the sequence of sampling intervals $\{\tau_k\}_{k\geqslant 0}$ for each aperiodic control strategy. The upper and lower edges of the outer box represent the maximum and minimum values, respectively. The upper and lower edges of the inner box represent the 25th and 75th percentile, respectively. Note that the plots have different sampling interval duration scales.

advantages of the corresponding event-triggered strategy. Simulation results confirm that using an EMH instead of a ZOH clearly improves the closed-loop performance in all cases. Note that the range where the `[12]-ZOH` strategy performs comparably to the other strategies is quite narrow.

In Figure 3, some statistics related with the sequence of sampling intervals $\{\tau_k\}_{k\geqslant 0}$ for each aperiodic control strategy are shown. We can see that the `[12]-ZOH` scheduling produces large and small sampling intervals throughout the simulation, while the `UR-{ZOH, EMH}` scheduling is confined to a smaller range of values. Comparing `UR-Event-EMH` and `UR-Self-EMH`, we can see that sampling intervals in `UR-Self-EMH` are smaller than the ones in `UR-Event-EMH`, demonstrating as previously mentioned that there is room for improvement in the derivation of the self-triggered implementation of the corresponding event-triggered strategy.

5. CONCLUSION

In this paper, we compared different approaches to the problem of self-triggered control of linear plants under bounded disturbances. We also developed a new self-triggered control strategy that represents a valuable alternative to the scheduling mechanism proposed in [15] and [16, 17]. The self-triggered state-feedback strategy introduced allows us to employ a model-based control architecture akin to that proposed in [19]. It was shown that the proposed control strategy renders the solutions of the closed-loop system GUUB that there exists a minimum time interval between

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sampling times, and a method for computing a lower bound on this minimum time is provided. An illustrative example shows through simulation results that the proposed self-triggered control strategy yields a clear gain in performance when compared to previous self-triggered strategies.

Concerning the state of the art in event-triggered control, future research will focus on self-triggered feedback based on measurements of plant outputs. The simulation results suggest that, for the moment, our proposed strategy is a valuable alternative to other self-triggered strategy when considering state-feedback problems and when one desires to obtain the benefits of using a model-based hold.

APPENDIX A: PROOFS

A.1. Proof of Theorem 1

According to (17) with the function h_{UR} defined as in (23), we have that, for all $k \geqslant 0$ and all $\delta \in [0, \tau_k]$,

$$h_{UR}(\delta, x_k) \leqslant 0 \Leftrightarrow R(\delta, V(x_k)) \geqslant U(\delta, \xi_{x_k}(\delta)) \geqslant V(x(t_k + \delta)), \tag{A.1}$$

where the last inequality follows from the definition of U (Section 3.5). Using the fact that $R(\delta, v) \leqslant R(0, v) = v$ for all $(\delta, v) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0}$, it follows from (A.1) that $V(x_k) \geqslant V(x(t_k + \delta))$ for all $\delta \in [0, \tau_k]$. In particular, this implies that $V(x_k) \geqslant V(x_{k+1})$. Using induction, we conclude that $V(x(t)) \leqslant V(x_0)$ for all $t \geqslant t_0$. Therefore, the set Ω_c where $c \geqslant V(x_0)$ is a positively invariant set, that is, $x_0 \in \Omega_c$ implies $x(t) \in \Omega_c$ for all $t \geqslant t_0$.

Next, we show that if $V(x_0) > W_\theta$, then there exists $T = T(V(x_0), W_\theta) \geqslant 0$ such that for $t \geqslant t_0 + T$, $V(x(t)) \leqslant W_\theta$ holds. First, we compute the derivative of U in order to δ. We have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\delta} U(\delta, \xi_x(\delta)) &= \frac{\partial U}{\partial \delta}(\delta, \xi_x(\delta)) + \frac{\partial U}{\partial x}(\delta, \xi_x(\delta)) \frac{\mathrm{d}}{\mathrm{d}\delta}\xi_x(\delta) && \text{(A.2)} \\ &= \frac{\partial M}{\partial \delta}(\delta, \xi_x(\delta)) + \left(\frac{\partial V}{\partial x}(\xi_x(\delta)) + \frac{\partial M}{\partial x}(\delta, \xi_x(\delta)) \right) \frac{\mathrm{d}}{\mathrm{d}\delta}\xi_x(\delta), && \text{(A.3)} \end{aligned}$$

where

$$\frac{\partial V}{\partial x}(x) = 2(Px)^\top \qquad \frac{\partial M}{\partial x}(\delta, x) = 2\beta(\delta)\frac{(P^2 x)^\top}{\|Px\|} \qquad \frac{\mathrm{d}}{\mathrm{d}\delta}\beta(\delta) = \varpi e^{\sigma\delta}$$

$$\frac{\partial M}{\partial \delta}(\delta, x) = 2\frac{\mathrm{d}}{\mathrm{d}\delta}\beta(\delta) \cdot (\|Px\| + \lambda_{\max}(P)\beta(\delta)),$$

and $\frac{\mathrm{d}}{\mathrm{d}\delta}\xi_x(\delta)$ is given by (22). Let $\tilde{x}(\delta, x)$ be defined as

$$\tilde{x}(\delta, x) = \xi_x(\delta) - \phi_x(\delta), \tag{A.4}$$

where $\phi_x \in \mathbb{R}^n$ satisfies, for all $\delta \geqslant 0$,

$$\frac{\mathrm{d}}{\mathrm{d}\delta}\phi_x(\delta) = \hat{A}\phi_x(\delta) + \hat{B}_1 K x, \quad \phi_x(0) = x, \tag{A.5}$$

for a given $x \in \mathbb{R}^n$. Note that $\hat{x}(t_k + \delta) = \phi_{x_k}(\delta)$ for all $k \geqslant 0$ and all $\delta \in [0, \tau_k)$. It follows that

$$\begin{aligned} \frac{\partial V}{\partial x}(\xi_x(\delta))\frac{\mathrm{d}}{\mathrm{d}\delta}\xi_x(\delta) &= 2\xi_x^\top(\delta)P[(A + B_1 K)\xi_x(\delta) - B_1 K\tilde{x}(\delta, x)] \\ &= -\xi_x^\top(\delta)Q\xi_x(\delta) + 2\xi_x^\top(\delta)PB_1 K\tilde{x}(\delta, x) \\ &\leqslant -\lambda_c V(\xi_x(\delta)) + 2\|P\xi_x(\delta)\|\|B_1 K\tilde{x}(\delta, x)\|. && \text{(A.6)} \end{aligned}$$

Evaluating (A.6) at $\delta = 0$ yields

$$\left.\frac{\partial V}{\partial x}(\xi_x(\delta))\frac{\mathrm{d}}{\mathrm{d}\delta}\xi_x(\delta)\right|_{\delta=0} \leqslant -\lambda_c V(x). \tag{A.7}$$

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The remaining terms in (A.2) evaluated at $\delta = 0$ are equal to

$$\left[\frac{\partial M}{\partial \delta}(\delta, \xi_x(\delta)) + \frac{\partial M}{\partial x}(\delta, \xi_x(\delta))\frac{\mathrm{d}}{\mathrm{d}\delta}\xi_x(\delta)\right]\Bigg|_{\delta=0} = 2\varpi \|Px\|. \tag{A.8}$$

The derivative of (23) is given by

$$\frac{\mathrm{d}}{\mathrm{d}\delta}h_{UR}(\delta, x) = \frac{\mathrm{d}}{\mathrm{d}\delta}U(\delta, \xi_x(\delta)) - \frac{\mathrm{d}}{\mathrm{d}\delta}R(\delta, V(x)), \tag{A.9}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}\delta}R(\delta, v) = \begin{cases} -\lambda_\vartheta R(\delta, v), & \text{if } v > W_\theta \\ 0, & \text{otherwise} \end{cases}. \tag{A.10}$$

Evaluating (A.9) at $\delta = 0$ and using the fact that $\|Px\| \leqslant \sqrt{\lambda_{\max}(P)V(x)}$ for all $x \in \mathbb{R}^n$, yields

$$\frac{\mathrm{d}}{\mathrm{d}\delta}h_{UR}(\delta, x)\Bigg|_{\delta=0} \leqslant -\lambda_c V(x) + 2\varpi\sqrt{\lambda_{\max}(P)V(x)} - \frac{\mathrm{d}}{\mathrm{d}\delta}R(\delta, V(x))\Bigg|_{\delta=0}. \tag{A.11}$$

Now, suppose $x_k \in \mathbb{R}^n$ is such that $V(x_k) > W_\theta$ (Figure A.1a). Then, (A.11) becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\delta}h_{UR}(\delta, x_k)\Bigg|_{\delta=0} &\leqslant -\lambda_c V(x_k) + 2\varpi\sqrt{\lambda_{\max}(P)V(x_k)} + \lambda_\vartheta V(x_k) \\ &= -\vartheta\lambda_c V(x_k) + 2\varpi\sqrt{\lambda_{\max}(P)V(x_k)} \\ &= -(\vartheta - \theta)\lambda_c V(x_k) - \theta\lambda_c V(x_k) + 2\varpi\sqrt{\lambda_{\max}(P)V(x_k)} \\ &\leqslant -(\vartheta - \theta)\lambda_c V(x_k) - \theta\lambda_c V(x_k) + 2\varpi\sqrt{\lambda_{\max}(P)V(x_k)} \\ &\leqslant -(\vartheta - \theta)\lambda_c V(x_k) < 0, \quad \text{for all } V(x_k) \geqslant W_\theta,\end{aligned} \tag{A.12}$$

where we have used the fact that $\theta < \vartheta$. Because for all $x_k \in \mathbb{R}^n$, we have $h_{UR}(0, x_k) = 0$, (A.12) implies that there exists $\tau_{\min,1} > 0$ such that $h_{UR}(\delta, x_k) \leqslant 0$ for all $\delta \in [0, \tau_{\min,1}]$. Moreover, when $t = t_{k+1}$, (17) and (23) imply that

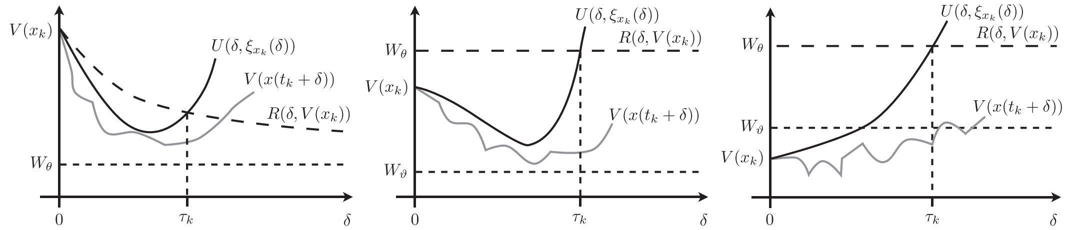
$$h_{UR}(\tau_k, x_{k+1}) \leqslant 0 \Leftrightarrow V(x_{k+1}) \leqslant R(\tau_k, V(x_k)) = V(x_k)e^{-\lambda_\vartheta \tau_k} \leqslant V(x_k)e^{-\lambda_\vartheta \tau_{\min,1}} < V(x_k). \tag{A.13}$$

If the initial condition is such that $V(x_0) > W_\theta$, applying (A.13) recursively, we conclude that

$$V(x_k) \leqslant V(x_0)e^{-\lambda_\vartheta(t_k - t_0)} \leqslant V(x_0)e^{-\lambda_\vartheta k \tau_{\min,1}}, \tag{A.14}$$

which implies that there exists a positive integer $N = N(x_0, \vartheta, \theta)$ such that $V(x(t)) \leqslant W_\theta$ for all $t \geqslant t_N$. In fact, we have that

$$N \geqslant \left\lceil \frac{1}{\lambda_\vartheta \tau_{\min,1}} \ln\left\{\frac{V(x_0)}{W_\theta}\right\}\right\rceil, \tag{A.15}$$



(a) $V(x_k) > W_\theta$. (b) $W_\vartheta < V(x_k) \leq W_\theta$. (c) $V(x_k) \leq W_\vartheta$.

Figure A.1. Possible triggering scenarios for different values of $V(x_k)$.

and $t_N \geqslant t_0 + N\tau_{\min,1}$. That is, the value of $V(x(t))$ is eventually smaller than or equal to W_θ. More formally, all trajectories starting in Ω_c enter the set Ω_{W_θ} in finite time. From here on, assume that $k \geqslant N$. At this point, two situations can occur.

If x_k is such that $W_\vartheta < V(x_k) \leqslant W_\theta$ (Figure A.1b), then $\frac{\mathrm{d}}{\mathrm{d}\delta} R(\delta, V(x_k))\big|_{\delta=0} = 0$ and (A.9) becomes

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}\delta} h_{UR}(\delta, x_k)\right|_{\delta=0} &\leqslant -\lambda_c V(x_k) + 2\varpi\sqrt{\lambda_{\max}(P)V(x_k)}\\ &\leqslant -(1-\vartheta)\lambda_c V(x_k) - \vartheta\lambda_c V(x_k) + 2\varpi\sqrt{\lambda_{\max}(P)V(x_k)}\\ &\leqslant -(1-\vartheta)\lambda_c V(x_k), \text{ for all } V(x_k) \geqslant W_\vartheta.\end{aligned} \tag{A.16}$$

This together with the fact that $h_{UR}(0, x_k) = V(x_0) - W_\theta \leqslant 0$, shows that there exists $\tau_{\min,2} > 0$ such that $h_{UR}(\delta, x_k) \leqslant 0$ for all $\delta \in [0, \tau_{\min,2}]$.

If x_k is such that $V(x_k) \leqslant W_\vartheta$ (Figure A.1c), then

$$\left.\frac{\mathrm{d}}{\mathrm{d}\delta} h_{UR}(0, x_k)\right|_{\delta=0} \leqslant -\lambda_c V(x_k) + 2\varpi\sqrt{\lambda_{\max}(P)V(x_k)} \leqslant 2\varpi\sqrt{\lambda_{\max}(P)W_\vartheta}, \tag{A.17}$$

that is, $\frac{\mathrm{d}}{\mathrm{d}\delta} h_{UR}(0, x_k)\big|_{\delta=0}$ is bounded. This together with the fact that

$$h_{UR}(0, x_k) = V(x_0) - W_\theta \leqslant W_\vartheta - W_\theta = W_1 \frac{\theta^2 - \vartheta^2}{\vartheta^2\theta^2} < 0, \tag{A.18}$$

implies that there exists $\tau_{\min,3} > 0$ such that $h_{UR}(\delta, x_k) \leqslant 0$ for all $\delta \in [0, \tau_{\min,3}]$.

In conclusion, the closed-loop system is uniformly ultimately bounded with ultimate bound $b = \sqrt{\kappa(P)}\frac{2\varpi}{\theta\lambda_c}$ and $T(a,b) = \frac{2}{\lambda_\vartheta \tau^*_{\min}} \ln\left\{\sqrt{\kappa(P)}\frac{a}{b}\right\}$ where $\kappa(P) = \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}$, and

$$\tau^*_{\min} = \min\{\tau_{\min,1}, \tau_{\min,2}, \tau_{\min,3}\} > 0, \tag{A.19}$$

is such that $t_{k+1} - t_k \geqslant \tau^*_{\min}$ for all $k \geqslant 0$. Moreover, because V is radially unbounded, the conclusion is global; that is, it holds for all initial conditions x_0 because for any $x_0 \in \mathbb{R}^n$, the constant c can be chosen large enough such that $x_0 \in \Omega_c$.

A.2. Proof of Lemma 1

Before giving the proof of Lemma 1 concerning the computation of a lower bound for $\tau^*_{\min}$, we require the following result.

Lemma 2

Consider the following linear ODE

$$\dot{y}(t) = Fy(t) + z(t), \tag{A.20}$$

where $y \in \mathbb{R}^n$, $y(t_0) = y_0$ and $z \in \mathbb{R}^n$ is a bounded function of time. Then, for all $t \geqslant t_0$, it holds that

$$\|y(t)\| \leqslant e^{\mu(F)(t-t_0)}\|y_0\| + \int_{t_0}^{t} e^{\mu(F)(t-s)}\|z(s)\|\,\mathrm{d}s. \tag{A.21}$$

Proof of Lemma 1

We will show that τ_H defined as (17) with $h = H$ is such that $\tau_{UR}(x) \geqslant \tau_H(x)$ for all $x \in \mathbb{R}^n$, and that $\tau_H(x) \geqslant \tau_{\min}$ for all $x \in \Omega_c$.

Applying Lemma 2 to (A.5) and using (A.22), implies that, for all $\delta \geqslant 0$,

$$\|\phi_x(\delta)\| \leqslant e^{\gamma_1 \delta}\|x\|, \tag{A.22}$$

where $\gamma_1 = \mu(\hat{A} + \hat{B}_1 K)$. We have from (A.4), (22), and (2a) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\delta}\tilde{x}(\delta, x) &= A\xi_x(\delta) + B_1 K\phi_x(\delta) - (\hat{A} + \hat{B}_1 K)\phi_x(\delta) \\ &= A\tilde{x}(\delta, x) + (\tilde{A} + \tilde{B}_1 K)\phi_x(\delta) \end{aligned}\tag{A.23}$$

where $\tilde{A} + \tilde{B}_1 K$ is equal to $A + B_1 K$ if a ZOH is used and equal to 0 if an EMH is used instead. Applying Lemma 2 to (A.23) with $y = \tilde{x}$, $F = A$, and $z = (\tilde{A} + \tilde{B}_1 K)\phi_x$ yields

$$\begin{aligned}\|\tilde{x}(\delta, x)\| &\leqslant \int_0^\delta e^{\gamma_2(\delta - s)} \left\|(\tilde{A} + \tilde{B}_1 K)\phi_x(s)\right\| \mathrm{d}s &\text{(A.24)}\\ &\leqslant \int_0^\delta e^{\gamma_2(\delta - s)}\|\tilde{A} + \tilde{B}_1 K\|e^{\gamma_1 s}\|x\|\mathrm{d}s &\text{(A.25)}\\ &\leqslant \frac{\|\tilde{A} + \tilde{B}_1 K\|}{\gamma_1 - \gamma_2}(e^{\gamma_1\delta} - e^{\gamma_2\delta})\|x\| &\text{(A.26)}\\ &\leqslant \frac{\|\tilde{A} + \tilde{B}_1 K\|}{\gamma_1 - \gamma_2}(e^{\gamma_1\delta} - e^{\gamma_2\delta})\sqrt{\frac{V(x)}{\lambda_{\min}(P)}} &\text{(A.27)}\\ &= E(\delta, V(x)), &\text{(A.28)}\end{aligned}$$

where $\gamma_2 = \mu(A)$. In the last inequality, we have used the fact that $\|x\| \leqslant \sqrt{\frac{V(x)}{\lambda_{\min}(P)}}$ for all $x \in \mathbb{R}^n$.

From (22) and (A.5), we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\delta}P\xi_x &= PA\xi_x + PB_1 K\phi_x &\text{(A.29)}\\ &= PAP^{-1}(P\xi_x) + PB_1 K\phi_x. &\text{(A.30)}\end{aligned}$$

Applying Lemma 2 to (A.29) with $y = P\xi_x$, $F = PAP^{-1}$, and $z = PB_1 K\phi_x$ and using (A.22) yields

$$\begin{aligned}\|P\xi_x(\delta)\| &\leqslant e^{\gamma_3\delta}\|Px\| + \int_0^\delta e^{\gamma_3(\delta - s)}\|PB_1 K\phi_x(s)\|\,\mathrm{d}s &\text{(A.31)}\\ &\leqslant e^{\gamma_3\delta}\|Px\| + \frac{\|PB_1 K\|}{\gamma_1 - \gamma_3}(e^{\gamma_1\delta} - e^{\gamma_3\delta})\|x\| &\text{(A.32)}\\ &\leqslant e^{\gamma_3\delta}\sqrt{\lambda_{\max}(P)V(x)} + \frac{\|PB_1 K\|}{\gamma_1 - \gamma_3}(e^{\gamma_1\delta} - e^{\gamma_3\delta})\sqrt{\frac{V(x)}{\lambda_{\min}(P)}} &\text{(A.33)}\\ &= \Xi(\delta, V(x)), &\text{(A.34)}\end{aligned}$$

where $\gamma_3 = \mu(PAP^{-1})$. In the last inequality, we have used the fact that $\|Px\| \leqslant \sqrt{\lambda_{\max}(P)V(x)}$ for all $x \in \mathbb{R}^n$.

Replacing (A.28) and (A.34) in (A.6) yields

$$\frac{\mathrm{d}}{\mathrm{d}\delta}V(\xi_x(\delta)) \leqslant -\lambda_c V(\xi_x(\delta)) + 2\Xi(\delta, V(x))\|B_1 K\|E(\delta, V(x)). \tag{A.35}$$

Applying standard comparison arguments, it follows that

$$V(\xi_x(\delta)) \leqslant G(\delta, V(x)), \tag{A.36}$$

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where the function G is defined in (31). Replacing (A.34) in (28), we obtain

$$M(\delta, \xi_x(\delta)) \leqslant \bar{M}(\delta, V(x)), \tag{A.37}$$

where the function $\bar{M}$ is defined in (32). For a given $x \in \mathbb{R}^n$ and for all $\delta \geqslant 0$, it holds

$$\begin{aligned} h_{UR}(\delta, \xi_x(\delta)) &= U(\delta, \xi_x(\delta)) - R(\delta, V(x)) &\text{(A.38)}\\ &\leqslant G(\delta, V(x)) + \bar{M}(\delta, V(x)) - R(\delta, V(x)) &\text{(A.39)}\\ &= H(\delta, V(x)). &\text{(A.40)}\end{aligned}$$

Note that $h_{UR}(0, x) = H(0, V(x)) = 0$. Hence, we have that $\tau_{UR}(x) \geqslant \tau_H(x)$ for all $x \in \mathbb{R}^n$. For a given $x \in \mathbb{R}^n$, it follows from (17) that

$$\tau_H(x) \geqslant \min\{\delta > 0 : H(\delta, V(x)) = 0\}. \tag{A.41}$$

Taking the minimum over all possible values of $V(x) \in [0, c]$ yields $\tau_H(x) \geqslant \tau_{\min}$ for all $x \in \Omega_c$, where $\tau_{\min}$ is given by (29).

We conclude by showing that $\tau_{\min}$ always exists. Computing the time derivative of $H(\delta, V(x))$ evaluated at $\delta = 0$ yields

$$\left.\frac{\mathrm{d}}{\mathrm{d}\delta} H(\delta, V(x))\right|_{\delta=0} = -\lambda_c V(x) + 2\varpi \sqrt{\lambda_{\max}(P) V(x)} - \left.\frac{\mathrm{d}}{\mathrm{d}\delta} R(\delta, V(x))\right|_{,\delta=0}, \tag{A.42}$$

which is exactly the expression in (A.11) that is used to show that $\tau^*_{\min}$ always exists. This implies that H satisfies sufficient conditions to guarantee the existence of $\tau_{\min}$. Furthermore, $\tau_{\min}$ is always finite because $\lim_{\delta \to +\infty} H(\delta, V(x)) = +\infty$ for all $x \in \mathbb{R}^n$. □

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