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ABSTRACT
This paper describes a solution to the problem of consensus/synchronisation for a general class of networked nonlinear multi-agent systems (MAS) using a distributed control strategy with an event-triggered communications (ETC) mechanism. We consider the case where the dynamics of each agent contain linear and Lipschitz nonlinear terms and the underlying communication graph is directed. The strategy proposed has two important properties: (i) it achieves practical consensus, i.e. the synchronisation error that measures the disagreement among the agents' states converges to a ball centred at the origin, with a radius that can be made arbitrarily small and (ii) the minimum of the inter-event times for each agent is lower bounded by a strictly positive number, hence Zeno behaviour is excluded. Furthermore, it affords system designers adequate tools to trade off the frequency of communications among agents against the level of performance achieved in MAS consensus/synchronisation. Numerical simulation results are also given.

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1. Introduction

The problem of consensus/synchronisation of multi agent systems (MAS) has been attracting tremendous interest in the last two decades due to its applications in the areas of sensor networks, mobile systems, autonomous robots, etc., where a group of agents must reach agreement on a final state (consensus) or trajectory (synchronisation). Some of the applications are described in Hung et al. (2020), Z. Li et al. (2010), and Rego et al. (2019), where consensus techniques were used to achieve desired geometric formations of multiple autonomous vehicles. For background materials on this topic the reader is referred to Olfati-Saber et al. (2007) and Z. Li and Duan (2015) where communications among the agents are assumed to occur continuously in time.

Driven by the fact that the bandwidth available for communications among multiple agents is severely limited in many practical applications, there has been a flurry of activity in the area of distributed event-triggered control and communications for multi-agent systems, as reported in Dimarogonas et al. (2012), Garcia et al. (2013), Meng and Chen (2013), Noorbakhsh and Ghaisari (2016), Nowzari and Cortés (2016), Nowzari et al. (2019) and the references therein. Among such studies, MAS with single integrator dynamics have received a great deal of attention, and many solutions for their coordination have been proposed, see for example the recent survey in Nowzari et al. (2019). One of earliest distributed event-triggered control solutions for MAS with single integrator dynamics was proposed in Dimarogonas et al. (2012), while solutions for both event-triggered control and communications can be found in Garcia et al. (2013) and Nowzari and Cortés (2016). The study of the coordination problem of MAS with double integrator dynamics using an event-triggered framework is addressed in Seyboth et al. (2013), which extends previous results for the case of continuous communications given in Ren and Beard (n.d.). More recently, a number of authors have addressed the problem of MAS coordination for the case where the agents have more general linear dynamics, Zhang et al. (2014), Zhu et al. (2014), Garcia et al. (2014, 2017), Almeida et al. (2017), and Hu et al. (2016). In Garcia et al. (2014) and Zhu et al. (2014), for example, the authors propose solutions for the MAS coordination problem where the triggering function adopted is dependent on the state, whereas an event-triggered mechanism that is dependent on time is proposed in Garcia et al. (2017) and Almeida et al. (2017).

Event triggered coordination of MAS with nonlinear dynamics has been less studied and only a few results that consider several particular classes of nonlinear system have appeared recently in the literature (Hung et al., 2019; H. Li et al., 2016; Liuzza et al., 2016; Su et al., 2016). A simple class of MAS with nonlinear dynamics was considered in Liuzza et al. (2016) and Hung et al. (2019), where the authors proposed a distributed model-based approach. The authors in Su et al. (2016) addressed the leader-following multi-agent systems consensus problem with an event-triggered control mechanism. An event-triggered sampling control approach for directed networks was studied in H. Li et al. (2016). In Hung et al. (2019), the authors proposed a solution to the synchronisation problem of one dimensional nonlinear MAS under weight-balanced directed graphs.

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Motivated by the above considerations, the main purpose of this paper is to develop a general framework for distributed control and event-triggered communication mechanism to solve the MAS consensus/synchronisation problem for which the agents dynamics and the network communication topologies are sufficiently general to address a large number of practical applications. The key benefits of the proposed strategy are twofold: (i) communication among agents occurs at discrete time instants rather than continuously, thus saving on bandwidth required for the MAS communication network and (ii) it affords designers a tool control the frequency of communications among agents while consensus/synchronisation of the MAS is still guaranteed. Compared with existing results in the literature, see for example Liuzza et al. (2016), Su et al. (2016), and Hung et al. (2019), the dynamics of the agents and the network topology are more general, i.e. the agents have linear and Lipschitz nonlinear dynamic terms, and the underlying communication graphs are directed. In this respect, our results extend those in Seyboth et al. (2013) and Hung et al. (2019). We also point out that with the proposed strategy, there exists a lower bound on minimum inter-event times for all agents, thus the strategy excludes Zeno behaviour.

The paper is organised as follows. Section 2 introduces the basic notation required and reviews important results from algebraic graph theory. The main problem is formulated in Section 3, whereas Section 4 presents the design of the ETC mechanism and the analysis of convergence of the resulting closed-loop MAS system. Section 5 shows how the proposed ETC mechanism extends existing results in the literature. Section 6 illustrates the performance of the proposed strategy through simulations. Finally, Section 7 contains some concluding remarks.

2. Preliminaries

2.1 Notation

In what follows, we let $\mathbb{R}, \mathbb{R}_{>0}$, and $\mathbb{R}_{\geq 0}$ denote the set of real, positive real, and nonnegative real numbers, respectively. We shall use the notation $\|\cdot\|$ to denote the Euclidean norm of a vector. We will use the notation $x(t^+) := \lim_{s\to t^+} x(s)$. Given matrices $A, B \in \mathbb{R}^{n\times n}$ the notation $A \succeq B$ implies that $A-B$ is positive semi-definite. A continuous function $\alpha : [0, a) \to [0,\infty)$ is said to be of class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to be of class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : [0, a) \times [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ is of class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$. Given a symmetric matrix A, the symbols $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the smallest and the largest eigenvalues of A.

2.2 Graph theory

A weight digraph $\mathcal{G} = \mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$ induced by the communication network of a multi-agent system consists of a set of N vertices (nodes) $\mathcal{V} = \{1, 2, \dots N\}$, a set of directed edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a weight adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$. The later satisfies the conditions $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. Here, self-edges (i, i) are not allowed and hence $a_{ii} = 0$. A path from vertex i to vertex j is an ordered sequence of vertices such that each immediate pair of the vertices is an edge. A digraph is strongly connected if there exists a path from any $i \in \mathcal{V}$ to any $j \in \mathcal{V}$. The set of in-neighbours and the set of out-neighbours of vertex i are defined as $\mathcal{N}_i^{\text{in}} = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$ and $\mathcal{N}_i^{\text{out}} = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$, respectively. The in- and out-degree matrices D^{in} and D^{out} are defined as $D^{\text{in}} = \text{diag}(d_i^{\text{in}})$ and $D^{\text{out}} = \text{diag}(d_i^{\text{out}})$ where

$$d_i^{\text{in}} = \sum_{j\in\mathcal{N}_i^{\text{in}}} a_{ij}, \quad d_i^{\text{out}} = \sum_{j\in\mathcal{N}_i^{\text{out}}} a_{ji},$$

respectively. A digraph is balanced if $D^{\text{in}} = D^{\text{out}}$. Any undirected graph is balanced. The Laplacian matrix L of a digraph is defined as $L = D^{\text{in}} - \mathcal{A}$. If $\mathcal{G}$ is strongly connected, then 0 is a simple eigenvalue of L with associated (right) eigenvector $\mathbf{1} := [1]_{N\times 1}$. Further, the digraph $\mathcal{G}$ is balanced if and only if $\mathbf{1}^{\mathrm{T}} L = \mathbf{0}$.

Remark 2.1: With the graph definition given above, we use the convention that an agent i can receive information from its neighbours in $\mathcal{N}_i^{\text{in}}$ and send information to its neighbours in $\mathcal{N}_i^{\text{out}}$.

The following lemma and definition will be used in the paper.

Lemma 2.1 ((Z. Li & Duan, 2015; Yu et al., 2010))**:** *Suppose that the graph $\mathcal{G}$ is strongly connected. Then, there is a positive left eigenvector $\mathbf{r} = [r_1, \dots, r_N]^{\mathrm{T}} \in \mathbb{R}^N$ of L associated with the zero eigenvalue of L s.t. $\mathbf{r}^{\mathrm{T}}\mathbf{1} = 1$ and $RL + L^{\mathrm{T}}R \succeq 0$, where $R = \text{diag}(r_1, \dots, r_N) \in \mathbb{R}^{N\times N}$.*

Definition 2.2 ((Generalised algebraic connectivity, (Yu et al., 2010))**):** Let L be the laplacian matrix of a strongly connected digraph $\mathcal{G}$. The generalised algebraic connectivity of the graph is defined as

$$a(L) = \min_{\mathbf{x}\neq\mathbf{0} \text{ and } \mathbf{x}\perp\mathbf{r}} \frac{\mathbf{x}^{\mathrm{T}}(RL + L^{\mathrm{T}}R)\mathbf{x}}{2\mathbf{x}^{\mathrm{T}}R\mathbf{x}}, \tag{1}$$

where R is defined in Lemma 2.1. With the above definition, and if the graph is balanced, $a(L) = \lambda_2(L_s)$, where $L_s \triangleq (L + L^{\mathrm{T}})/2$. For undirected graphs, $a(L) = \lambda_2(L)$, where the later is called the Fiedler eigenvalue of the graph.

3. Problem formulation

We consider the problem of synchronising the trajectories of multiple networked nonlinear systems (agents). We denote by $\mathbf{x}_i \in \mathbb{R}^n$ and $\mathbf{u}_i \in \mathbb{R}^m$ the state and the input of agent i, respectively. Each agent has the nonlinear dynamics given by

$$\dot{\mathbf{x}}_i = A\mathbf{x}_i + \mathbf{f}(\mathbf{x}_i, t) + B\mathbf{u}_i, \tag{2}$$

for all $i \in \mathcal{V}$, where A, B have appropriate dimensions. We assume the nonlinear map $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is piecewise continuous in t and Lipschitz in $\mathbf{x}$ with Lipschitz constant $l \in \mathbb{R}_{\geq 0}$, that is, for any $\mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, $\|\mathbf{f}(\mathbf{y}, t) - \mathbf{f}(\mathbf{z}, t)\| \leq l\|\mathbf{y} - \mathbf{z}\|$.

We denote by $\mathcal{G}$ the digraph that describes the inter-agent communications topology and assume that $\mathcal{G}$ is strongly connected. Due to the communication constraints imposed by $\mathcal{G}$, each agent is only able to receive information from its in-neighbouring agents. The consensus/synchronisation problem that we study consists of finding a distributed protocol for $\mathbf{u}_i = \mathbf{u}_i(\mathbf{x}_i, \mathbf{x}_j, t); j \in \mathcal{N}_i^{\text{in}}, i \in \mathcal{V}$ such that the agents reach consensus asymptotically, that is, $\mathbf{x}_1(t) = \mathbf{x}_2(t) = \cdots = \mathbf{x}_N(t)$ as $t \to \infty$ and the agents remain synchronised with identical dynamics described by $\dot{\mathbf{x}}_i = A\mathbf{x}_i + \mathbf{f}(\mathbf{x}_i, t)$ for all $i \in \mathcal{V}$ as $t \to \infty$. From (2), this implies that under these circumstances the input $\mathbf{u}_i(t) \to \mathbf{0}$ as $t \to \infty$ for all $i \in \mathcal{V}$.

It was shown in Z. Li et al. (2012) and Z. Li and Duan (2015) that the distributed protocol given by

$$\mathbf{u}_i = cK \sum_{j \in \mathcal{N}_i^{\text{in}}} a_{ij}(\mathbf{x}_i - \mathbf{x}_j) = cK \sum_{j=1}^{N} a_{ij}(\mathbf{x}_i - \mathbf{x}_j) \tag{3}$$

for all $i \in \mathcal{V}$, with proper choices of c and K, solves the consensus/synchronisation problem. However, the protocol given by (3) relies on continuous communications among the agents. This, in turn, requires that the in-neighbours of agent i transmit their states to agent i continuously to update the input $\mathbf{u}_i$. Unfortunately, in practice the communication bandwidth might be limited as all agents might share the same network. This motivated the development of the distributed control scheme with an ETC mechanism described in this paper with the purpose of reducing the number of messages exchanged and the frequency of communications among the agents. These characteristics are of the utmost importance in applications where the transmission medium imposes stringent communication constraints (e.g. cooperative control of multiple autonomous underwater vehicles (AUVs) (Rego et al., 2019)).

4. Consensus/synchronisation with event-triggered communications

In this section, we first describe the process of designing an ETC mechanism, after which we perform an analysis of the convergence properties of MAS consensus/synchronisation.

4.1 Design of the ETC mechanism

In an ETC mechanism, the control law (3) uses, for each agent, the estimates of its in-neighbour states ($\mathbf{x}_j; j \in \mathcal{N}_i^{\text{in}}$), instead of their true states. Let $\hat{\mathbf{x}}_j^i$ be an estimate of $\mathbf{x}_j$ computed by agent i (the procedure to compute this estimate will be explained later). The control law with the ETC mechanism that we propose is given by

$$\mathbf{u}_i = cK \sum_{j \in \mathcal{N}_i^{\text{in}}} a_{ij}(\mathbf{x}_i - \hat{\mathbf{x}}_j^i) \tag{4}$$

for all $i \in \mathcal{V}$. The underlying idea in the proposed ETC mechanism is that if $\hat{\mathbf{x}}_j^i$ can provide a 'good' estimate of $\mathbf{x}_j$, then the communication among agents does not have to be continuous. We propose the following estimator for $\hat{\mathbf{x}}_j^i$:

$$\hat{\mathbf{x}}_j^i : \begin{cases} \dot{\hat{\mathbf{x}}}_j^i = A\hat{\mathbf{x}}_j^i + \mathbf{f}(\hat{\mathbf{x}}_j^i, t), & t \in [t_{ij,k}, t_{ij,k+1}), \\ \hat{\mathbf{x}}_j^i(t_{ij,k}^+) = \mathbf{x}_j(t_{j,k}) \end{cases} \tag{5}$$

for $i \in \mathcal{V}$ and $j \in \mathcal{N}_i^{\text{in}}$, where $\{t_{j,k}\}_{k \in \mathbb{N}}$ is the sequence of time instants at which agent j broadcasts its state to its out-neighbours, while $\{t_{ij,k}\}_{k \in \mathbb{N}}$ is the corresponding sequence of time instants at which agent i receives the update on the state of agent $j; j \in \mathcal{N}_i^{\text{in}}$. The structure of the estimator (5) is motivated by the fact that if consensus/synchronisation were achieved perfectly, i.e. $\mathbf{x}_i = \mathbf{x}_j$ for all $i, j \in \mathcal{V}$, then the input of each agent would remain at zero, and in this case the estimated variables would be the true states of the agents.

In order to control the error between $\mathbf{x}_j$ and $\hat{\mathbf{x}}_j^i$ we define a variable $\hat{\mathbf{x}}_j; j \in \mathcal{V}$ as a 'replica' of $\hat{\mathbf{x}}_j^i; i \in \mathcal{N}_j^{\text{out}}$ at agent j. The dynamics of $\hat{\mathbf{x}}_j$ are given by

$$\hat{\mathbf{x}}_j : \begin{cases} \dot{\hat{\mathbf{x}}}_j = A\hat{\mathbf{x}}_j + \mathbf{f}(\hat{\mathbf{x}}_j, t), & t \in [t_{j,k}, t_{j,k+1}) \\ \hat{\mathbf{x}}_j(t_{j,k}^+) = \mathbf{x}_j(t_{j,k}); \end{cases} \tag{6}$$

Clearly, if communication delays are negligible, i.e. $t_{ij,k} = t_{j,k}$ for all k, then it can be seen from (5) and (6) that $\hat{\mathbf{x}}_j(t) = \hat{\mathbf{x}}_j^i(t)$ for all t. See Figure 1 as an illustration of the underlying idea of the ETC mechanism for a network with three agents.Since $\hat{\mathbf{x}}_j(t) = \hat{\mathbf{x}}_j^i(t)$, the protocol in (4) can be rewritten as

$$\mathbf{u}_i = cK \sum_{j \in \mathcal{N}_i^{\text{in}}} a_{ij}(\mathbf{x}_i - \hat{\mathbf{x}}_j). \tag{7}$$

Because $a_{ij} = 0$ if $j \notin \mathcal{N}_i^{\text{in}}$, Equation (7) can be rewritten as

$$\mathbf{u}_i = cK \sum_{j=1}^{N} a_{ij}(\mathbf{x}_i - \mathbf{x}_j + \mathbf{e}_j) \tag{8}$$

for all $i, j \in \mathcal{V}$, where

$$\mathbf{e}_j \triangleq \mathbf{x}_j - \hat{\mathbf{x}}_j \tag{9}$$

for all $j \in \mathcal{V}$. Compared with the protocol for continuous communications given by (3), the protocol in (8) has the contribution of the estimation error $\mathbf{e}_j$ for all $j \in \mathcal{V}$. The key point in the proposed ETC mechanism is that if $\mathbf{e}_j$ for all $j \in \mathcal{V}$ can be enforced to be bounded then, as we will show later, the synchronisation error between agents will also be bounded. This

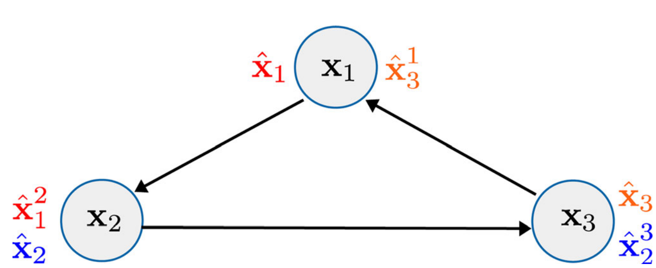

Figure 1. An illustrative example: without communication delays, the ETC mechanism ensures that $\hat{\mathbf{x}}_j$ and $\hat{\mathbf{x}}_j^i$ are synchronised, i.e. $\hat{\mathbf{x}}_j(t) = \hat{\mathbf{x}}_j^i(t)$ for all t and $i \in \mathcal{N}_j^{\text{out}}, j \in \mathcal{V}$.

set of ideas borrows from the groundbreaking work presented in Tabuada (2007), where the author introduced an event-triggered control mechanism for the stabilisation problem of nonlinear systems. To ensure that the estimation error $\mathbf{e}_j;\ j \in \mathcal{V}$ is bounded, we allow agent $j;\ j \in \mathcal{V}$ to broadcast its state ($\mathbf{x}_j$) whenever $\|\mathbf{e}_j\|$ reaches a designed bounded *threshold function* $h_j(\cdot)$ that, in general, can be parameterised by time as $h_j(t)$. Formally, for each agent $i; i \in \mathcal{V}$ we define an event-triggering function $\delta_i(t)$ for the communications as

$$\delta_i(t) = \|\mathbf{e}_i(t)\| - h_i(t), \tag{10}$$

where $h_i(t)$ belongs to a class of non-negative functions $\mathcal{C}$ defined by $\mathcal{C} := \{f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} | c_l \leq f(t) \leq c_u\}$ for all $i \in \mathcal{V}$. For example, $h_i(t) = c_1 + c_2 e^{-\alpha t}$, where $c_1, c_2, \alpha \in \mathbb{R}_{\geq 0}$ are constant parameters, see Seyboth et al. (2013), Hung et al. (2019), and Almeida et al. (2017). With the above definition, agent $i;\ i \in \mathcal{V}$ will transmit its state to its out-neighbours whenever $\delta_i(t) \geq 0$.

At this point, we have the necessary ingredients to summarise the proposed ETC framework for the consensus/synchronisation problem described above. The resulting procedure is summarised in Algorithm 1.

Algorithm 1 ETC mechanism for agent i

```
At every time t, agent i implements the following procedure:
procedure COORDINATION AND COMMUNICATION
    if δ_i(t) ≥ 0 where δ_i(t) is computed using (10), then
        Broadcast x_i(t);
        Reset x̂_i using (6);
    end if
    if Receive a new message from agent j then
        if j ∈ N_i^in then
            Reset x̂_j^i using (5);
        end if
    end if
    Run the estimators (5) and (6);
    Update the protocol for u_i using (4);
return u_i
end procedure
```

4.2 Convergence analysis

We now analyse the consensus/synchronisation properties of the closed-loop MAS with the ETC algorithm given in Algorithm 1. To this end, let

$$\boldsymbol{\xi}_i \triangleq \mathbf{x}_i - \sum_{j=1}^{N} r_j \mathbf{x}_j, \tag{11}$$

where r_j is the jth component of the positive left eigenvector $\mathbf{r}$ of the graph Laplacian matrix defined in Lemma 2.1. Note that if the graph is connected and balanced, then $\mathbf{r} = \mathbf{1}/N$. In this case, $\boldsymbol{\xi}_i$ measures the disagreement between agent i's state and the average of all agents' states. Let also $\boldsymbol{\xi} \triangleq [\boldsymbol{\xi}_1^{\mathrm{T}}, \dots, \boldsymbol{\xi}_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{nN}$

Algorithm 2 Selecting the control gains in (4)

```
procedure CHOSE K
    Solve the following linear matrix inequality (LMI) for
    variables P, τ, and μ:
```

$$P = P^{\mathrm{T}}, P \succ 0, \tag{14a}$$

$$\tau > 0, \quad \mu > 0, \tag{14b}$$

$$\begin{bmatrix} AP + PA^{\mathrm{T}} - 2\tau BB^{\mathrm{T}} + \mu l I_n \& P \\ P \& - \mu I_n / l \end{bmatrix} \prec 0 \tag{14c}$$

```
    Chose matrix K = −B^T P^{−1} .
end procedure
procedure CHOSE c
    Compute a(L) given by (1).
    Chose any coupling gain c ≥ τ/a(L).
end procedure
```

be the synchronisation error vector that captures the disagreement among the agents' states. From (11), $\boldsymbol{\xi}$ can be rewritten as

$$\boldsymbol{\xi} = W\mathbf{x}, \tag{12}$$

where $\mathbf{x} \triangleq [\mathbf{x}_1^{\mathrm{T}}, \dots, \mathbf{x}_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{nN}$ and

$$W \triangleq (I_N - \mathbf{1}\mathbf{r}^{\mathrm{T}}) \otimes I_n. \tag{13}$$

With the above definition, it is clear that all agents are synchronised, that is, $\mathbf{x}_1 = \mathbf{x}_2 = \cdots = \mathbf{x}_N$ if and only if $\boldsymbol{\xi} = \mathbf{0}$. Therefore, to analyse the synchronisation of the MAS, we analyse the convergence of the synchronisation error $\boldsymbol{\xi}$ to zero. Before proceeding to the main result of the paper, we assume that the gain c and K in protocol (4) can be computed using Algorithm 2. The rationale behind this algorithm will become clear in the next sub-section.

Theorem 4.1: *Consider the closed-loop multi-agent system described by* (2)*, driven by the distributed control strategy and the ETC mechanism specified in Algorithm* 1*. Suppose further that there exists a solution to the LMI system* (14a)–(14c)*. Let* $\mathbf{h} \triangleq [h_1, \dots, h_N]^{\mathrm{T}}$ *be the vector-valued function containing all threshold functions. Then, the following statements hold true.*

(i) There exist a $\mathcal{KL}$ *class function* β *and a* $\mathcal{K}$ *class function* γ *such that for any initial state* $\boldsymbol{\xi}(t_0)$ *the synchronisation error* $\boldsymbol{\xi}$ *satisfies*

$$\|\boldsymbol{\xi}(t)\| \leq \beta(\|\boldsymbol{\xi}(t_0)\|, t - t_0) + \gamma\left(\sup_{t_0 \leq \tau \leq t} \|\mathbf{h}(\tau)\|\right). \tag{15}$$

(ii) If the threshold functions are designed such that $\lim_{t\to\infty} h_i(t) = c_l$ *for all* $i \in \mathcal{V}$*, then*

$$\lim_{t\to\infty} \|\boldsymbol{\xi}(t)\| \leq r_1 = \frac{2\|F_1\|\lambda_{\max}(R \otimes P^{-1})}{\lambda_{\min}(H_2)\lambda_{\min}(R \otimes P^{-1})}\sqrt{N}c_l, \tag{16}$$

where

$$F_1 \triangleq (R \otimes P^{-1})W(\mathcal{A} \otimes cBB^{\mathrm{T}}P^{-1}) \tag{17}$$

and

$$H_2 = -(I_N \otimes P)(R \otimes H_1)(I_N \otimes P), \tag{18}$$

with

$$H_1 \triangleq AP + PA^{\mathrm{T}} - 2\tau BB^{\mathrm{T}} + \mu l I_n + \mu^{-1} l P^2 \tag{19}$$

(iii) *If all threshold functions h_i; $i \in \mathcal{V}$ are lower bounded by $c_l > 0$, then the closed-loop MAS system does not exhibit Zeno behaviour.*

Proof: (i) To show the inequality in (15) we analyse the closed-loop dynamics of the synchronisation error vector $\boldsymbol{\xi}$. First, substituting in (2) the ETC protocol for $\mathbf{u}_i$ given by (8) yields

$$\dot{\mathbf{x}} = (I_N \otimes A + cL \otimes BK)\mathbf{x} + \mathbf{g}(\mathbf{x}, t) + \boldsymbol{\eta}, \tag{20}$$

where

$$\mathbf{g}(\mathbf{x}, t) \triangleq \begin{bmatrix} \mathbf{f}(\mathbf{x}_1, t) \\ \vdots \\ \mathbf{f}(\mathbf{x}_N, t) \end{bmatrix}, \quad \boldsymbol{\eta} \triangleq \begin{bmatrix} cBK \sum_{j=1}^{N} a_{1j}\mathbf{e}_j \\ \vdots \\ cBK \sum_{j=1}^{N} a_{Nj}\mathbf{e}_j \end{bmatrix}. \tag{21}$$

Thus, from (12) and (20) the closed-loop dynamics of $\boldsymbol{\xi}$ are given by

$$\begin{aligned} \dot{\boldsymbol{\xi}} &= W\dot{\mathbf{x}} \\ &= (I_N \otimes A + cL \otimes BK)\boldsymbol{\xi} + W\left(\mathbf{g}(\mathbf{x}, t) + \boldsymbol{\eta}\right), \end{aligned} \tag{22}$$

where we used the fact that $W(I_N \otimes A + cL \otimes BK) = (I_N \otimes A + cL \otimes BK)W$. We now consider the Lyapunov function candidate defined as

$$V = \boldsymbol{\xi}^{\mathrm{T}}(R \otimes P^{-1})\boldsymbol{\xi}, \tag{23}$$

where P is given by Algorithm 2 and R is defined in Lemma 2.1. Since $P \succ 0$ and R is a positive diagonal matrix, V is clearly positive definite. Its time derivative along the trajectory of (22) is given by

$$\dot{V} = \underbrace{2\boldsymbol{\xi}^{\mathrm{T}}(R \otimes P^{-1})(I_N \otimes A + cL \otimes BK)\boldsymbol{\xi}}_{\triangleq X} + \underbrace{2\boldsymbol{\xi}^{\mathrm{T}}(R \otimes P^{-1})W\mathbf{g}(\mathbf{x}, t)}_{\triangleq Y} + \underbrace{2\boldsymbol{\xi}^{\mathrm{T}}(R \otimes P^{-1})W\boldsymbol{\eta}}_{\triangleq Z} \tag{24}$$

The term X in (24) can be rewritten as

$$X = 2\boldsymbol{\xi}^{\mathrm{T}}(R \otimes P^{-1}A + cRL \otimes P^{-1}BK)\boldsymbol{\xi}. \tag{25}$$

Let $\boldsymbol{\zeta} \triangleq (I_N \otimes P^{-1})\boldsymbol{\xi}$. Substituting $K = -B^{\mathrm{T}}P^{-1}$ defined in Algorithm 2 in (25) yields

$$\begin{aligned} X &= \boldsymbol{\zeta}^{\mathrm{T}}\left(R \otimes (AP + PA^{\mathrm{T}})\right)\boldsymbol{\zeta} \\ &\quad - \boldsymbol{\zeta}^{\mathrm{T}}\left(c(RL + L^{\mathrm{T}}R) \otimes BB^{\mathrm{T}}\right)\boldsymbol{\zeta}. \end{aligned} \tag{26}$$

Because $(\mathbf{r}^{\mathrm{T}} \otimes I_n)\boldsymbol{\xi} = \mathbf{0}$, $(\mathbf{r}^{\mathrm{T}} \otimes I_n)\boldsymbol{\zeta} = \mathbf{0}$. Therefore, using Definition 2.2, we obtain

$$\boldsymbol{\zeta}^{\mathrm{T}}\left(c(RL + L^{\mathrm{T}}R) \otimes BB^{\mathrm{T}}\right)\boldsymbol{\zeta} \geq 2\boldsymbol{\zeta}^{\mathrm{T}}(R \otimes ca(L)BB^{\mathrm{T}})\boldsymbol{\zeta}.$$

Thus, X in (26) is bounded as

$$X \leq \boldsymbol{\zeta}^{\mathrm{T}}\left(R \otimes (AP + PA^{\mathrm{T}} - 2ca(L)BB^{\mathrm{T}})\right)\boldsymbol{\zeta}. \tag{27}$$

Furthermore, if c is chosen such that $c \geq \tau/a(L)$ with $\tau > 0$, as given by Algorithm 2, then

$$X \leq \boldsymbol{\zeta}^{\mathrm{T}}\left(R \otimes (AP + PA^{\mathrm{T}} - 2\tau BB^{\mathrm{T}}\right)\boldsymbol{\zeta}. \tag{28}$$

Let $\bar{\mathbf{x}} \triangleq \sum_{i=1}^{N} r_i\mathbf{x}_i$. The term Y in (24) can be expanded as

$$\begin{aligned} Y &= 2\sum_{i=1}^{N} r_i\boldsymbol{\xi}_i^{\mathrm{T}}P^{-1}\left(\mathbf{f}(\mathbf{x}_i, t) - \sum_{j=1}^{N} r_j\mathbf{f}(\mathbf{x}_j, t)\right) \\ &= \underbrace{2\sum_{i=1}^{N} r_i\boldsymbol{\xi}_i^{\mathrm{T}}P^{-1}\left(\mathbf{f}(\mathbf{x}_i, t) - \mathbf{f}(\bar{\mathbf{x}}, t)\right)}_{\triangleq Y_1} \\ &\quad + \underbrace{2\sum_{i=1}^{N} r_i\boldsymbol{\xi}_i^{\mathrm{T}}P^{-1}\left(\mathbf{f}(\bar{\mathbf{x}}, t) - \sum_{j=1}^{N} r_j\mathbf{f}(\mathbf{x}_j, t)\right)}_{\triangleq Y_2}. \end{aligned}$$

Because $\sum_{i=1}^{N} r_i\boldsymbol{\xi}_i = 0$, it follows that $Y_2 = 0$. Thus, using the Lipschitz assumption on $\mathbf{f}(\cdot)$ and Young's inequality, we obtain

$$\begin{aligned} Y = Y_1 &\leq 2\sum_{i=1}^{N} r_i l\left\|\boldsymbol{\xi}_i^{\mathrm{T}}P^{-1}\right\|\|\boldsymbol{\xi}_i\| \\ &\leq \sum_{i=1}^{N} r_i l\boldsymbol{\xi}_i^{\mathrm{T}}\left(\mu(P^{-1})^2 + \mu^{-1}I_n\right)\boldsymbol{\xi}_i \\ &= \boldsymbol{\zeta}^{\mathrm{T}}\left(R \otimes \left(\mu l I_n + \mu^{-1}lP^2\right)\right)\boldsymbol{\zeta} \end{aligned} \tag{29}$$

for every $\mu > 0$.

We now compute the upper bound for the term Z in (24). Defining $\mathbf{e} = [\mathbf{e}_1^{\mathrm{T}}, \ldots, \mathbf{e}_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{nN}$, $\boldsymbol{\eta}$ can be rewritten from (21) as

$$\boldsymbol{\eta} = (\mathcal{A} \otimes cBK)\mathbf{e} = -(\mathcal{A} \otimes cBB^{\mathrm{T}}P^{-1})\mathbf{e}, \tag{30}$$

where $\mathcal{A}$ is the adjacency matrix of the graph $\mathcal{G}$. Substituting (30) to Z in (24) we obtain

$$\begin{aligned} Z &\leq 2\|\boldsymbol{\xi}\|\left\|(R \otimes P^{-1})W(\mathcal{A} \otimes cBB^{\mathrm{T}}P^{-1})\right\|\|\mathbf{e}\| \\ &= 2\|\boldsymbol{\xi}\|\|F_1\|\|\mathbf{e}\|, \end{aligned} \tag{31}$$

where F_1 is given by (17). Thus, from (28), (29) and (31) the time derivative of V in (24) is upper bounded as

$$\dot{V} \leq \boldsymbol{\zeta}^{\mathrm{T}}(R \otimes H_1)\boldsymbol{\zeta} + 2\|\boldsymbol{\xi}\|\|F_1\|\|\mathbf{e}\|, \tag{32}$$

where H_1 is given by (19). Note that H_1 is symmetric. Furthermore, from the LMI (14c) in Algorithm 2, H_1 is also negative definite (this can be seen by using Schur's complement to rewrite H_1 in the form of LMI (14c)). Because $\boldsymbol{\zeta} = (I_N \otimes P^{-1})\boldsymbol{\xi}$, $\boldsymbol{\xi} = (I_N \otimes P)\boldsymbol{\zeta}$. Inserting $\boldsymbol{\xi}$ and H_2 defined in (18) in (32), we obtain

$$\begin{aligned}\dot{V} &\leq -\boldsymbol{\xi}^{\mathrm{T}} H_2 \boldsymbol{\xi} + 2\|\boldsymbol{\xi}\|\|F_1\|\|\mathbf{e}\| \\ &\leq -\lambda_{\min}(H_2)\|\boldsymbol{\xi}\|^2 + 2\|\boldsymbol{\xi}\|\|F_1\|\|\mathbf{e}\| \\ &\leq -\alpha_{\xi}(\|\boldsymbol{\xi}\|) \quad \forall \|\boldsymbol{\xi}\| \geq \rho(\|\mathbf{e}\|) \end{aligned} \tag{33}$$

where $\alpha_{\xi} \in \mathcal{K}$ and $\rho \in \mathcal{K}_{\infty}$ are functions defined by $\alpha_{\xi}(r) = (1-\theta)\lambda_{\min}(H_2)r^2$ and $\rho(r) = \frac{2\|F_1\|}{\theta\lambda_{\min}(H_2)}r$, respectively, and $\theta \in (0,1)$. Invoking Theorem 4.19 in Khalil (2002), we conclude that V, given by (23), is an ISS-Lyapunov function for the synchronisation error vector system given by (22) and the system is input to state stable (ISS) with respect to the state $\boldsymbol{\xi}$ and the input $\mathbf{e}$. This implies that there exist functions $\beta \in \mathcal{KL}, \gamma \in \mathcal{K}$ such that for any initial state $\boldsymbol{\xi}(t_0)$ the synchronisation error vector satisfies

$$\|\boldsymbol{\xi}(t)\| \leq \beta(\|\boldsymbol{\xi}(t_0)\|, t-t_0) + \gamma\left(\sup_{t_0 \leq \tau \leq t} \|\mathbf{e}(\tau)\|\right) \tag{34}$$

for all $t \geq t_0$. Furthermore, with the ETC mechanism the error $\mathbf{e}_i$ satisfies $\|\mathbf{e}_i(t)\| \leq h_i(t)$ for all $t \geq t_0$, hence $\|\mathbf{e}(t)\| \leq \|\mathbf{h}(t)\|$ for all $t \geq t_0$. Substituting this result in (34), we conclude that the inequality (15) holds true.

(ii) We now prove the second statement in the theorem. For this purpose, we define two class $\mathcal{K}$ functions $\alpha_1(\|\boldsymbol{\xi}\|) = \lambda_{\min}(R \otimes P^{-1})\|\boldsymbol{\xi}\|^2$ and $\alpha_2(\|\boldsymbol{\xi}\|) = \lambda_{\max}(R \otimes P^{-1})\|\boldsymbol{\xi}\|^2$. Clearly, the Lyapunov function in (23) satisfies $\alpha_1(\|\boldsymbol{\xi}\|) \leq V \leq \alpha_2(\|\boldsymbol{\xi}\|)$ for all $\boldsymbol{\xi}$. Hence, using again Theorem 4.19 in Khalil (2002) we obtain

$$\gamma(r) = \alpha_1^{-1}(\alpha_2(\rho(r))) = \frac{2\|F_1\|\lambda_{\max}(R \otimes P^{-1})}{\theta\lambda_{\min}(H_2)\lambda_{\min}(R \otimes P^{-1})}r. \tag{35}$$

Furthermore, if $\lim_{t\to\infty} h_i(t) = c_l$ for all $i \in \mathcal{V}$, then $\lim_{t\to\infty} \|\mathbf{h}(t)\| = \sqrt{N}c_l$. Substituting this relation and (35) in (15) we conclude that the synchronisation error $\boldsymbol{\xi}$ satisfies (16).

(iii) We now prove the third statement of the theorem by showing that the minimum inter-event time for every agent is strictly positive if $c_l > 0$. To this end, let $t_{i,k}$ and $t_{i,k+1}$ be successive triggering times at which agent i sends its state to its out-neighbouring agents. We consider the evolution of the estimation error $\mathbf{e}_i(t)$ during the interval $\mathcal{T}_{i,k} \triangleq [t_{i,k}, t_{i,k+1})$ when $\mathbf{e}_i(t)$ is continuous. It follows from (9) that $\dot{\mathbf{e}}_i(t) = \dot{\mathbf{x}}_i - \dot{\hat{\mathbf{x}}}_i = A\mathbf{e}_i + (\mathbf{f}(\mathbf{x}_i, t) - \mathbf{f}(\hat{\mathbf{x}}_i, t)) + B\mathbf{u}_i$. Furthermore, $\mathbf{e}_i(t_{i,k}^+) = 0$ because $\hat{\mathbf{x}}_i(t_{i,k}^+) = \mathbf{x}_i(t_{i,k}^+)$ (see (6)). Hence,

$$\begin{aligned}\|\mathbf{e}_i(t)\| &\leq \int_{t_{i,k}}^{t} \|A\mathbf{e}_i(\tau)\| \,\mathrm{d}\tau \\ &\quad + \int_{t_{i,k}}^{t} \left\|\mathbf{f}(\mathbf{x}_i(\tau), \tau) - \mathbf{f}(\hat{\mathbf{x}}_i(\tau), \tau)\right\| \mathrm{d}\tau \\ &\quad + \int_{t_{i,k}}^{t} \|B\mathbf{u}_i(\tau)\| \,\mathrm{d}\tau \\ &\leq \int_{t_{i,k}}^{t} (\|A\| + l)\|\mathbf{e}_i(\tau)\| \,\mathrm{d}\tau + \int_{t_{i,k}}^{t} \|B\mathbf{u}_i(\tau)\| \,\mathrm{d}\tau \end{aligned} \tag{36}$$

for all $t \in \mathcal{T}_{i,k}$. Note that the last inequality follows from the Lipschitz property of $\mathbf{f}(\cdot)$. To find an upper bound for $\mathbf{e}_i$ we now compute an upper bound for $\mathbf{u}_i(t)$ for all $i \in \mathcal{V}$. Let $\mathbf{u} := [\mathbf{u}_1^{\mathrm{T}}, \ldots, \mathbf{u}_N^{\mathrm{T}}]^{\mathrm{T}}$. Substituting $K = -B^{\mathrm{T}}P^{-1}$ to (8) we obtain

$$\begin{aligned}\mathbf{u} &= -(cL \otimes BB^{\mathrm{T}}P^{-1})\mathbf{x} - (\mathcal{A} \otimes BB^{\mathrm{T}}P^{-1})\mathbf{e} \\ &= -G\boldsymbol{\xi} - T\mathbf{e}, \end{aligned} \tag{37}$$

where $G \triangleq (cL \otimes BB^{\mathrm{T}}P^{-1})$ and $T \triangleq (\mathcal{A} \otimes BB^{\mathrm{T}}P^{-1})$. Observe that for all $i \in \mathcal{V}$

$$\|\mathbf{u}_i(t)\| \leq \|\mathbf{u}(t)\| \overset{(37)}{\leq} \|G\|\|\boldsymbol{\xi}(t)\| + \|T\|\|\mathbf{e}(t)\|. \tag{38}$$

Recall also that $\|\mathbf{e}(t)\| \leq \|\mathbf{h}(t)\| \leq \sqrt{N}c_u$ for all $t \geq t_0$, where c_u is the upper bound for $h_i(t)$. Hence, it follows from (15) and (38) that

$$\|\mathbf{u}_i(t)\| \leq \bar{u} \triangleq \|G\|\left(\beta(\|\boldsymbol{\xi}(t_0)\|, 0) + \gamma(\sqrt{N}c_u)\right) + \|T\|\sqrt{N}c_u$$

for al $t \geq t_0$. Since $\beta \in \mathcal{KL}$, $\bar{u}$ only depends on the initial condition of $\boldsymbol{\xi}$, and $\bar{u}$ is an upper bound for $u_i(t)$ for all $t \geq t_0$ and $i \in \mathcal{V}$. Therefore, from (36), $\mathbf{e}_i(t)$ can be bounded as

$$\|\mathbf{e}_i(t)\| \leq \int_{t_{i,k}}^{t} (\|A\| + l)\|\mathbf{e}_i(\tau)\| \,\mathrm{d}\tau + (t - t_{i,k})\|B\|\bar{u}.$$

Let

$$\lambda(t) = (t - t_{i,k})\|B\|\bar{u}, \qquad c = \|A\| + l. \tag{39}$$

Applying Gronwall-Bellman inequality (see Lemma A.1 in Khalil, 2002) to the last equality, we obtain

$$\|\mathbf{e}_i(t)\| \leq \lambda(t) + c\int_{t_{i,k}}^{t} \lambda(s)e^{c(t-s)} \,\mathrm{d}s.$$

Integration the rightmost term by parts yields

$$\begin{aligned}\|\mathbf{e}_i(t)\| &\leq \lambda(t) - \lambda(s)e^{c(t-s)}\Big|_{t_{i,k}}^{t} + \|B\|\bar{u}\int_{t_{i,k}}^{t} e^{c(t-s)} \,\mathrm{d}s \\ &= \underbrace{\|B\|\bar{u}\left(e^{(\|A\|+l)(t-t_{i,k})} - 1\right) / (\|A\| + l)}_{\triangleq \Delta(t)}. \end{aligned} \tag{40}$$

Because a broadcast event for agent i is triggered if and only if $\delta_i(t)$ crosses zero or $\|\mathbf{e}_i(t)\| = h_i(t)$, the next event is triggered not earlier than time $t^* > t_{i,k}$, given by the solution of the equation $\Delta(t) = c_l$. Hence, the minimum inter-event time for any agent is lower bounded by

$$\tau_1 := t^* - t_{i,k} = \frac{\ln\left(1 + c_l(\|A\| + l)/(\|B\|\bar{u})\right)}{\|A\| + l} > 0. \tag{41}$$

Since there is a positive lower bound τ_1 on the inter-event intervals, there are no accumulation points in the event sequences and therefore Zeno behaviour is excluded. This completes the proof of Theorem 4.1. ■

The result stated in (i) indicates that the synchronisation error vector is input-to state stable (ISS) with respect to the input $\mathbf{h}$ (see the definition of ISS in Khalil (2002)). This also implies that the synchronisation error is bounded for any bounded threshold functions that, in the context of the ETC mechanism, are user designed functions used as tuning knobs to trade off communication rate against synchronisation performance. The result in (ii) is a consequence of (i) and indicates that if the threshold functions converge, then the synchronisation error vector converges as well. The size of the ball that the synchronisation error $\boldsymbol{\xi}$ converges to depends explicitly on the lower bound of the threshold functions. That is, the asymptotic bound in (16) can be made arbitrarily small by decreasing c_l. Note also that continuous communication is a special case of the ETC mechanism where the triggering threshold function $h_i(t) = 0$ for all t and $i \in \mathcal{V}$.

Remark 4.1: Note that the LMI (14a)–(14c) is equivalent to the LMI in Z. Li et al. (2012) (see Equation (6) in Z. Li et al., 2012). The feasibility of the LMI in (14) was discussed in Z. Li et al. (2012). For the case of linear MAS systems or when the Lipschitz constant $l = 0$, the LMI in (14) is equivalent to finding $\tau > 0$ and $P \succ 0$ such that $AP + PA^{\mathrm{T}} - 2\tau BB^{\mathrm{T}} \prec 0$. The feasibility of the inequalities is equivalent to the pair (A, B) being controllable.

Remark 4.2: In Algorithm 2 the computation of c requires global knowledge about the agents' communication network (embodied in the generalised algebraic connectivity of the agents' graph, denoted $a(L)$) but the computation of K requires only knowledge about the dynamics of the agents themselves. This implies that only c might need to be computed in a distributed manner. In the present paper, because the main focus of the work is on the proposed event-triggered communication mechanism, for simplicity of exposition, we assumed that the agents' network topology is known to all agents in advance. Therefore, c can be computed in advance for all agents. However, if this assumption is not satisfied, then for each agent the gain c can be replaced by c_{ij} which, in this case, is viewed as considered as a time-varying variable that is updated in a distributed fashion using a similar consensus strategy for the input $\mathbf{u}_i$. For details on this type approach, we refer the reviewer to the work of Z. Li et al. (2013).

5. Extensions and unified results

In the previous section, the distributed protocol given by (4) uses the true state of agent i ($\mathbf{x}_i$) to update $\mathbf{u}_i$. In this section, we analyse the synchronisation of MAS by using the following protocol

$$\mathbf{u}_i = cK \sum_{j \in \mathcal{N}_i^{\text{in}}} a_{ij}(\hat{\mathbf{x}}_i - \hat{\mathbf{x}}_j^i), \tag{42}$$

where $\hat{\mathbf{x}}_i$ and, $\hat{\mathbf{x}}_j^i$ are given by given by (6) and (5), respectively, for all $i, j \in \mathcal{V}$. As explained in the previous section, without communication delays (42) can be rewritten as

$$\mathbf{u}_i = cK \sum_{j \in \mathcal{N}_i^{\text{in}}} a_{ij}(\hat{\mathbf{x}}_i - \hat{\mathbf{x}}_j). \tag{43}$$

At this point, the question may arise regarding the intuition behind the protocol defined by (42), in comparison with (4). While the motivation behind (4) is clear from the design process of the proposed ETC mechanism, the protocol (42) is motivated by one of the earliest works in the field of event-triggered control and communications where consensus of multiple one dimensional single integrator systems was studied (Dimarogonas et al., 2012). In the latter, the protocol (42) facilitates the derivation of a so-called 'state-dependent' triggering threshold function that plays a role similar to that of the triggering threshold function h_i (see (10)). In the literature (see for example Almeida et al., 2017; Hung et al., 2019; Nowzari et al., 2019) the triggering threshold function used in our present paper is called 'time-dependent'. Compared with the 'state-dependent' type of threshold, the 'time-dependent' type is simpler to design and implement, especially for MASs that have complex dynamics and modelled by directed graphs (Hung et al., 2019; Nowzari et al., 2019).

With the protocol (42), we will see that the result of our ETC mechanism in this section generalises some of the existing results in the literature (for example Hung et al., 2019; Seyboth et al., 2013) where the agents' dynamics and the network topologies are special cases of the problem considered in the current paper. We obtain the following result.

Theorem 5.1: *Consider the closed-loop MAS described by* (2), *driven by the distributed control strategy and the ETC mechanism specified in Algorithms 1, where the distributed protocol given by* (42) *is used rather than* (4). *Assume further that the LMI* (14) *in Algorithm 2 is feasible. Then, the following statements hold true.*

(i) *The properties* (i) *and* (iii) *stated in Theorem 4.1 are satisfied.*
(ii) *Furthermore, if the threshold functions are designed such that* $\lim_{t\to\infty} h_i(t) = c_l$ *for all* $i \in \mathcal{V}$, *then*

$$\lim_{t\to\infty} \|\boldsymbol{\xi}(t)\| \leq r_2 = \frac{2\|F_2\|\lambda_{\max}(R \otimes P^{-1})}{\lambda_{\min}(H_2)\lambda_{\min}(R \otimes P^{-1})}\sqrt{N}c_l, \tag{44}$$

where

$$F_2 \triangleq (R \otimes P^{-1})W(L \otimes cBB^{\mathrm{T}}P^{-1}), \tag{45}$$

and H_2 *is given by* (18).

Proof: The proof is done similarly to the proof of Theorem 1. ■

It can be observed from (16) and (44) that r_1 is computed using the adjacency matrix $\mathcal{A}$, whereas r_2 is computed using the laplacian matrix L (see the difference between F_1 and F_2). Basically, the two bounds are equivalent as both matrices represent the connectivity of the considered digraph.

We next show that the asymptotic bound given by (44) generalises some existing results in the literature.

5.1 Average consensus (Seyboth et al., 2013)

In Seyboth et al. (2013), the authors considered the average consensus problem for undirected graphs where the dynamics of each agent is given by

$$\dot{\mathbf{x}}_i = \mathbf{u}_i \tag{46}$$

for all $i \in \mathcal{V}$ and $\mathbf{x}_i, \mathbf{u}_i \in \mathbb{R}$. Compared with the general form given by (2), the system (46) has $n = m = 1$, $A = 0$, $B = 1$ and $l = 0$. With these parameters, the LMI given by (14) is feasible for every $\tau > 0$ and $P > 0$. Thus, we can chose $\tau = \lambda_2(L) > 0$, $c = 1$ and $P = 1$ to satisfy Algorithm (2). As a result, $K = -B^{\mathrm{T}}P^{-1} = -1$.

Note also that with the system (46) the estimator for each $j; j \in \mathcal{V}$ in (6) becomes $\dot{\hat{\mathbf{x}}}_j = \mathbf{0}$, i.e. $\hat{\mathbf{x}}_j(t) = \mathbf{x}_j(t_{j,k})$ for all $t \in [t, t_{j,k+1})$. Since the graph is undirected with $a_{ij} = 1$ for all $i, j \in \mathcal{V}$, we can let $\mathcal{N}_i \triangleq \mathcal{N}_i^{\text{in}} = \mathcal{N}_i^{\text{out}}$. With the above parameters, the control law in (43) becomes

$$\mathbf{u}_i = -\sum_{j \in \mathcal{N}_i} (\hat{\mathbf{x}}_i - \hat{\mathbf{x}}_j), \tag{47}$$

which is identical to the control law used in Seyboth et al. (2013) (see Equation (4) in Seyboth et al., 2013). Furthermore, because the graph considered is undirected, it follows from Lemma 2.1 that $R = I_N/N$. Substituting the above parameters in H_2 given by (18) and in F_2 given by (45) we obtain $H_2 = 2\lambda_2(L)I_N$ and $F_2 = L$. In addition, the authors use the threshold functions $h_i(t) = c_1 e^{-\alpha t} + c_0$ for all $i \in \mathcal{V}$, hence $c_l = c_0$. Inserting the above parameters in (44) we obtain

$$r_2 = \|L\|\sqrt{N}c_0/\lambda_2(L), \tag{48}$$

which is identical to the bound in Theorem 3.2 (see Equation (8) in Seyboth et al., 2013).

5.2 *A simple nonlinear system (Hung et al., 2019)*

In Hung et al. (2019), the authors considered a synchronisation problem for balanced digraphs where the dynamics of each agent are given by

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, t) + \mathbf{u}_i \tag{49}$$

for all $i \in \mathcal{V}$ and $\mathbf{x}_i, \mathbf{u}_i \in \mathbb{R}$. Compared with the general form given by (2) the system in (49) has $n = m = 1$, $A = 0$, $B = 1$. In this case, the LMI given by (14) is equivalent to $\tau, \mu, P > 0$ and $-2\tau + \mu l + \mu^{-1} l P^2 < 0$. By choosing $P = 1$ and $\mu = 1$ any $\tau > l$ will satisfy the LMI. Since the digraph is balanced, $a(L) = \lambda_2(L_s)$, where L_s is defined in Definition 1. Furthermore, by taking $c = \tau/a(L) = \tau/\lambda_2(L_s) > l/\lambda_2(L_s)$ so as to satisfy Algorithm 2, this choice of c also satisfies the gain condition given by Theorem 1 in Hung et al. (2019). Substituting the above parameters in H_2 given by (18) and in F_2 given by (45) we obtain $H_2 = 2(c\lambda_2(L_s) - l)I_N$ and $F_2 = cL$. Inserting the above parameters in (44) we obtain

$$r_2 = c\|L\|\sqrt{N}c_l/(c\lambda_2(L_s) - l), \tag{50}$$

which is identical to the bound given by Corollary 1 in Hung et al. (2019)). Notice that the coupling gain c in (50) plays the same role as k in Hung et al. (2019).

6. Simulation examples

This section illustrates the performance of the proposed ETC mechanism using computer simulations. We consider a network of six agents whose topology is modelled by the digraph illustrated in Figure 2. Let $\mathbf{x}_i = [x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4}]^{\mathrm{T}} \in \mathbb{R}^4$ be the state of agent i. The dynamics of each agent are given by (2) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & -1 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 1.95 & 0 & -1.95 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}},$$

and $\mathbf{f}(\mathbf{x}_i, t) = [0 \ \ 0 \ \ 0 \ \ -0.333\sin(x_{i3})]^{\mathrm{T}}$. Therefore, the corresponding globally Lipchitz constant for $\mathbf{f}(\cdot)$ is $l = 0.333$. Running Algorithm 2 using the CVX tool box (Grant & Boyd,

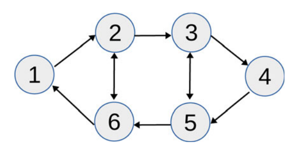

Figure 2. Example. Network topology.

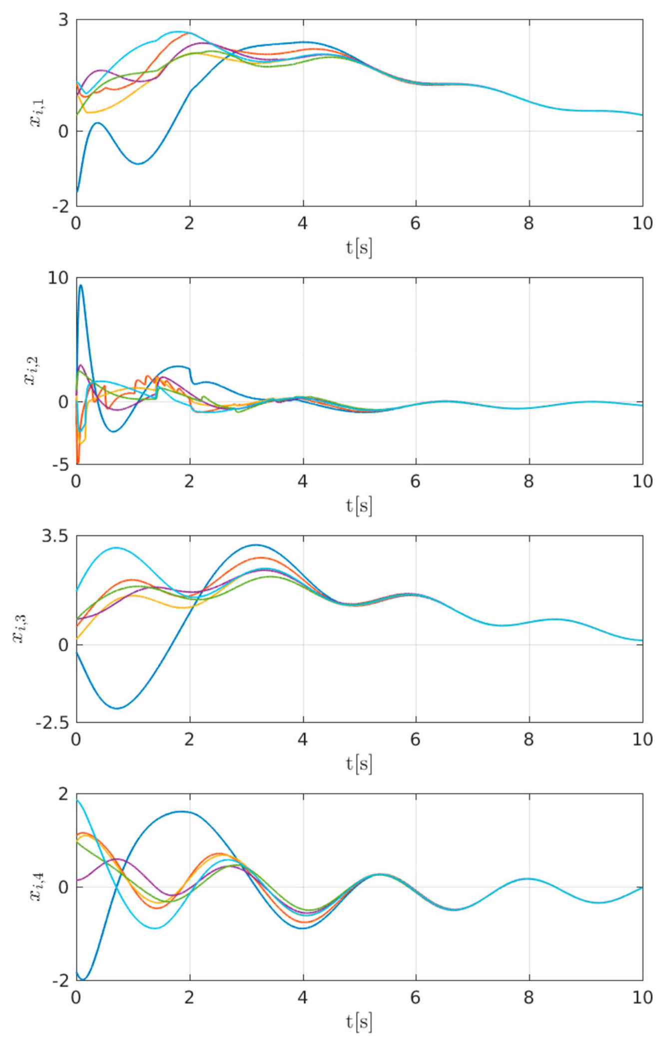

Figure 3. ETC mechanism with controller (4). The synchronisation of the agents' states.

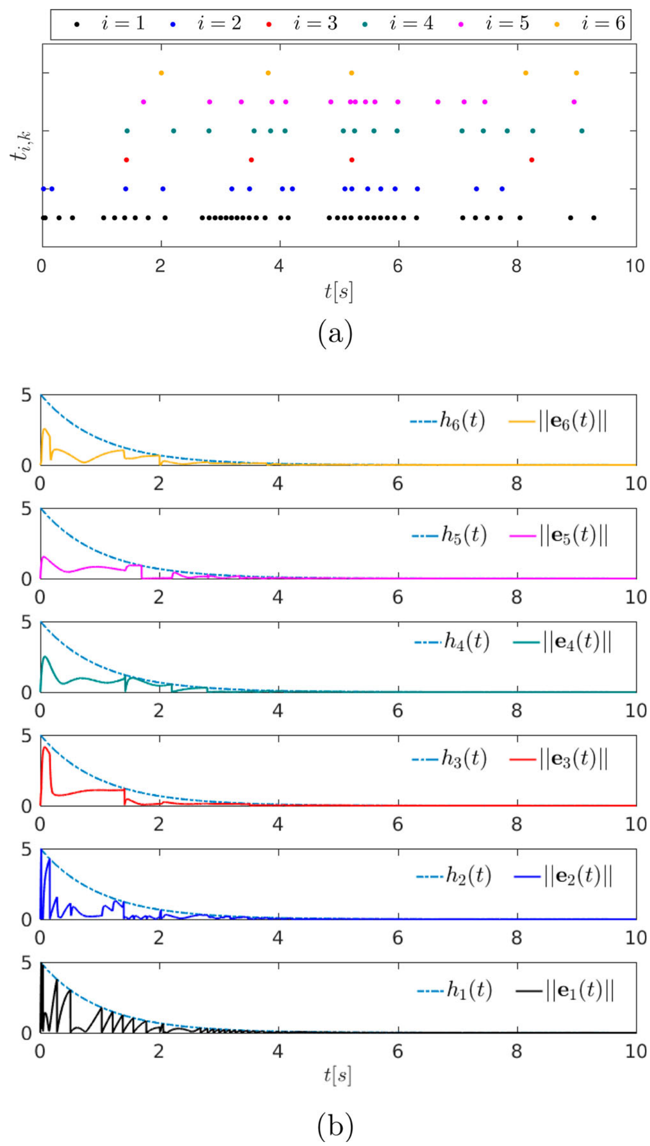

Figure 4. ETC mechanism with controller (4). (a) Sequence of broadcast time instants for each agent. (b) Evolution of the estimation errors $\|\mathbf{e}_i(t)\|$ and the threshold triggering functions $h_i(t), i = 1, \ldots, 6$.

2014), we obtain $c = 2251$ and $K = [-0.0470, -0.0134, 0.0285, -0.0224]$. For the threshold functions, we set $h_i(t) = c_0 + c_1 e^{-c_2 t}$ for all $i \in \mathcal{V} := \{1, \ldots, 6\}$, with $c_0 = 1e{-}3, c_1 = 5$, and $c_2 = 1$.

Figures 3 and 4 show the simulation results with the ETC mechanism using the control law given by (4). It can be observed in Figure 3 that the states of all agents are synchronised asymptotically, i.e. the agents' states reach consensus and evolve along a common trajectory. Figure 4 shows that the agents broadcast their states at discrete time instants for which the estimation errors $\|\mathbf{e}_i(t)\|$ hit the triggering threshold functions $h_i(t); i \in \mathcal{V}$. The minimum inter-event time and the number of broadcast events for each agent during the simulation period are shown in Table 1.

Figure 5 shows the trajectory of the synchronisation error with continuous communications (C-C), and with the ETC mechanism using controllers (4) and (42). According to Theorem 4.1, using controller (4) the asymptotic bound for

Table 1. Minimum inter-event times and number of events.

Agent i	$\min\{t_{i,k+1} - t_{i,k}\}(s)$	Total number of events
$i = 1$	0.029	42
$i = 2$	0.117	16
$i = 3$	2.103	4
$i = 4$	0.189	15
$i = 5$	0.081	15
$i = 6$	0.854	5

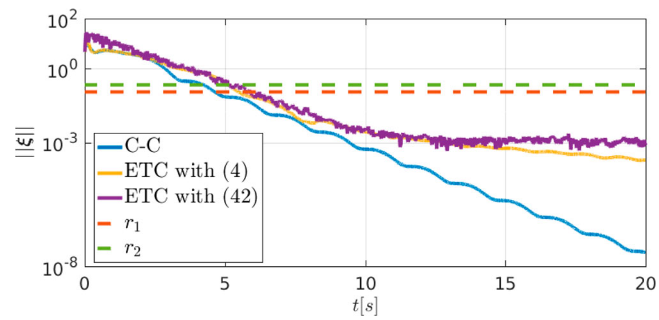

Figure 5. Example. Trajectories of the synchronisation error $\|\xi\|$ and asymptotic bounds with different communication mechanisms.

the synchronisation error is computed using (16) as $r_1 = 0.1102$. Using controller (42), the asymptotic bound is computed using (44) as $r_2 = 0.2139$. The figure clearly shows that asymptotically, the synchronisation error is upper bounded by the asymptotic bounds. Recall that the asymptotic bounds can be made arbitrarily small by tuning the lower bound on the threshold functions. Hence, the synchronisation error would get closer to zero if c_0, the lower bound of the triggering function were reduced. This also would make the synchronisation error closer to that obtained using continuous communications. However, this would potentially make communications among the agents more frequent.

7. Conclusions

This paper described a general event-triggered communication framework for consensus/synchronisation of nonlinear multi-agent systems. The agent's dynamics and the network topology considered are sufficiently general enough to address a large number of applications. We showed how consensus and synchronisation can be achieved and how communications among agents can be reduced by using the proposed event-triggered communication mechanism. Further, the minimum inter-event time for every agent was shown to be strictly positive, thus excluding the occurrence of Zeno behaviour. Future work will aim at extending the ETC mechanism to heterogeneous multi agent nonlinear system and taking into account package losses and communication delays.

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