Robot Formations Motion Dynamics

Based on Scalar Fields

Andrés G. García **Pedro U. Lima**

Instituto de Sistemas e Robótica,

Instituto Superior Técnico

Av. Rovisco Pais, 1 1049-001

Lisboa, Portugal

e-mail: {andres,pal}@isr.ist.utl.pt

Abstract

Non-holonomic systems may appear in several forms, including combinations between holonomic and non-holonomic constraints for vehicle formations. Examples of the latter are non-holonomic formation constraints with holonomic vehicles or holonomic formation constraints with non-holonomic vehicles.

In this paper the problem of non-holonomic systems with holonomic or non-holonomic constraints is addressed by reformulating the problem using scalar fields. This has the advantage of leading to a definition of force which allows to formulate the motion of a team of non-holonomic vehicles in matrix form, both for holonomic and non-holonomic constraints. Furthermore, the constraints can be systematically included in this formulation.

1 Introduction

A relevant issue in cooperative robotics is to develop a tool to solve the trajectories of a robot team given the mathematical model of the robots kinematics (holonomic or non-holonomic) [1], [2] and the constraints between the team robots [3].

The idea is to find the possible trajectories of the team for a given task; the task could be expressed as constraints among the robots themselves (e.g., to transport large objects) and the constraints for the movement of the team (a desired trajectory with desired velocities).

Up to now several approaches have been made considering the formation as a rigid body [4], [5]. In [1] the authors consider a set of planar robots manipulating a flexible object and develop a controller for the entire formation.

Furthermore, obstacle avoidance is very important, especially for cluttered environments. Several approaches in the literature approach this problem by modeling the environment as stochastic [6]. In the case of known structured environments one of the more useful technique is the potential fields [7], [8]; according to that technique the obstacles are modelled as a fields (scalar fields) which generate attractive forces in order to avoid a collision.

Any physical model based approach to robot navigation with obstacle avoidance is rooted in the definition of force (interaction). This could be in the form of Newton's laws or, more generally, in the form of Lagrange's equations.

Lagrange's equations provide a generalization of Newton's laws, which allows the representation of a particle system (in this case a robot team) independently of the coordinate framework used [9, pp 360-362]. This has the advantage that we can formulate every problem related to team formations in adequate coordinates. Moreover, the formulation of Lagrange allows the modeling of the motion of a given team as a partial differential equation system where the boundary conditions come in the form of constraint equations over the position and velocity of every robot.

From this point of view the constraints are such that if some position coordinate of any robot stands time invariant, the number of partial differential equations in the system mentioned above is reduced. This is known as a holonomic problem. The general case where no position coordinate stands constant is known as a non-holonomic problem.

The non-holonomic problem has the hard inconvenient that in order to solve the trajectories of the entire team we need to solve the coupled system of partial differential equations without the possibility to reduce the order of the system.

On the other hand, Lagrange's equations still keep the vectorial nature of Newton's laws except for conservative cases [9, pp 35-39] where scalar fields can be used for the model.

This paper combines the advantages of using scalar fields and the concept of time-space interval [10, pp 3-9] leading to a novel definition of force which permits the formulation of the motion of a team in a matrix form both for holonomic and non-holonomic cases. A method that includes the constraints in a systematic way will be introduced in this paper.

2 Theoretical Background

A review of the most relevant math concepts for this work is required before we proceed. In this section some concepts of advanced calculus frequently used in the sequel will be reviewed. For further review of related linear algebra see [11].

System of Partial differential equations:

Given a system of partial differential equations, an important problem is to determine when there is a solution, i.e., if there exists a function which verifies the partial differential equations in the system. This is solved by the result for analytical functions known as the Cauchy-Kowalewsky theorem [12].

Theorem 1:

Let $t, x^1 \cdots x^{m-1}$ be coordinates in $\Re^m$. Consider a system of n partial differential equations for n unknown functions $\phi_1, \cdots, \phi_n$ in $\Re^m$, having the form:

$$\frac{\partial^2 \phi_i}{\partial t^2} = F_i(t, x^\alpha, \phi_j, \frac{\partial \phi_j}{\partial t}, \frac{\partial \phi_j}{\partial x^\alpha}, \frac{\partial^2 \phi_j}{\partial t \partial x^\alpha}, \frac{\partial^2 \phi_j}{\partial x^\beta \partial x^\alpha}), \quad (1)$$

where the notation x^α means the equivalence $x^\alpha = [x^1, \cdots, x^{m-1}]$. On the other hand the functions F_i are analytical functions of their variables. Let $f_i(x^\alpha)$ and $g_i(x^\alpha)$ be analytic functions, then there is an open neighbourhood O of the hyper surface $t=t_0$ such that within O there exists an unique analytic solution of the equation (1) with the initial conditions:

$$\phi_i(t_0, x^\alpha) = f_i(x^\alpha), \quad \frac{\partial \phi}{\partial t}(t_0, x^\alpha) = g_i(x^\alpha) . \quad (2)$$

3 Team formations Dynamics

The main idea behind this paper is to develop a new model of physical interaction (forces in the Newton's context) which handles team motion simpler than Newton's laws or, which is the same, Lagrange's equations.

We need a linear model to apply to determine the interaction and the final motion for the system. In Newton's definition, forces are quantities that can be added up in a linear manner. Because of that, it is always simple to formulate Newton's laws but not so easy to solve the final motion of a team.

To start with our development we can consider another important aspect, which took a lot of consideration in the past; this aspect takes into account the fact that Newtonian mechanics always requires an inertial frame in order to write correctly the motion equations. While solving this inconvenient, it was found that the invariant quantity with the transformation of coordinates is the one named *space-time interval* [10, pp 3-9].

In this context we want to introduce a novel definition in order to cover the problem of the invariance between coordinate transformations and the non-holonomic problem introduced in Section 1, using scalar fields, which are adequate for computational calculus. In the sequel we will consider the number of robots in the team to be equal to *N*.

Definition 1:

A single interaction within a given team is defined by the following space-time law:

$$k \cdot t^2 = \phi(X, t) + l \cdot \psi(X, p_1, p_2, \cdots p_N), \quad (3)$$

where X is the position vector of the entire system represented by the position of each robot $X = [x_1, y_1, z_1, \cdots, z_N]$, ϕ is the term due to the external and internal interactions and ψ is the term due to the team itself, related with the properties p_i of the team. In the case of mechanical systems we will see that the properties are the masses m_i but in the general case we can set other properties of interest. The constants k and l are real numbers that adapt units. Finally t is the time variable.

k and l will be calculated for any particular problem with adequate initial conditions. Before, we need to determine the mathematical form of ψ and to verify that the definition agrees with Newton's laws, that is, we need to find the relation between the forces and our fields ϕ.

We can perform this task applying total temporal derivatives to (3):

$$2\cdot k\cdot t=\dot{\phi}+l\cdot\sum_{i=1}^{N}\Big(\frac{\partial\psi(p_1,p_2,\cdots,p_N)}{\partial x_i}\cdot\dot{x}_i+\frac{\partial\psi(p_1,p_2,\cdots,p_N)}{\partial y_i}\cdot\dot{y}_i+\frac{\partial\psi(p_1,p_2,\cdots,p_N)}{\partial z_i}\cdot\dot{z}_i\Big),\quad(4)$$

where the temporal dependence of the functions ϕ could be defined as follows:

$$k\cdot t^2-\phi(X,t)-l\cdot\psi(X,p_1,p_2,\cdots p_N)=0\Leftrightarrow G(X,t)=0\ .(5)$$

The function G represents (3) in a more compact manner in order to apply the implicit function theorem:

$$\frac{\partial G(X,t)}{\partial t}=0\Rightarrow 2\cdot k\cdot t-\frac{\partial\phi}{\partial t}=0\Rightarrow 2\cdot k\cdot t=\frac{\partial\phi}{\partial t},\quad(6)$$

where (6) was obtained with the goal of getting a matrix null space representation in the sequel.

Recalling that the goal of our definition is to get a relation between Newton's forces and our fields ϕ, this suggests that the term $M\cdot\ddot{X}$ should appear in the derivation where M is the inertia matrix of the team defined by $M=\begin{bmatrix} m_1\cdot I_3 & \cdots & \Theta \\ \vdots & \ddots & \vdots \\ \Theta & \cdots & m_N\cdot I_3\end{bmatrix}$ and I_3 is a diagonal matrix living in $\Re^{3x3}$ with ones in the main diagonal and zeros elsewhere.

By definition from (4) and (6):

$$\begin{aligned}&2\cdot K\cdot t=\frac{\partial\phi}{\partial x^{\alpha}}\cdot\dot{X}+\frac{\partial\phi}{\partial t}+\\&+l\cdot\frac{\partial\psi(p_1,p_2,\cdots,p_N)}{\partial x^{\alpha}}\cdot\dot{X}\Rightarrow\\&\frac{\partial\phi}{\partial x^{\alpha}}\cdot\dot{X}=-l\cdot\frac{\partial\psi(p_1,p_2,\cdots,p_N)}{\partial x^{\alpha}}\cdot\dot{X}\end{aligned}\quad(7)$$

Now taking the partial derivates in (7) with respect to the spatial coordinates:

$$0=\frac{\partial^2\phi}{\partial x^{\alpha}\partial x^{\beta}}\cdot\dot{X}+l\cdot\frac{\partial^2\psi(p_1,p_2,\cdots,p_N)}{\partial x^{\alpha}\partial x^{\beta}}\cdot\dot{X}\ ,\quad(8)$$

To obtain (8) is used the fact $\left.\frac{\partial\dot{X}}{\partial x^{\alpha}}\right|_{\dot{X}}=0$. It should be:

$$\frac{\partial^2\psi(p_1,p_2,\cdots,p_N)}{\partial x^{\alpha}\partial x^{\beta}}=M\ ,\quad(9)$$

Solving (9) we obtain:

$$\psi=\frac{1}{2}\cdot X'\cdot M\cdot X\ ,\quad(10)$$

where X' indicates the transpose of X.

Note: In the calculus of ψ we took the second derivative in (7) because we want to get *symmetric fields* ϕ, which clearly demands a quadratic form for ψ.

Finally our dynamic law for team formations takes the form:

$$k\cdot t^2=\phi(X,t)+\frac{1}{2}\cdot l\cdot X'\cdot M\cdot X\ .\quad(11)$$

3.1 The Method

Let us now use Definition 1 in order to get a method to avoid the difficulties found in the Lagrange's equations for non-holonomic systems. We will also show how this definition yields an interpretation for the number of possible solutions in a matrix context.

We first rewrite (8) as follows:

$$\begin{aligned}&(J(\phi)+l\cdot M)\cdot\dot{X}=0\\&J(\phi)=\begin{bmatrix}\frac{\partial^2\phi}{\partial x_1^{\ 2}} & \cdots & \frac{\partial^2\phi}{\partial x_1\partial z_N}\\ \vdots & \ddots & \vdots\\ \vdots & \cdots & \frac{\partial^2\phi}{\partial z_N^{\ 2}}\end{bmatrix},\end{aligned}\quad(12)$$

Notice (12) yields an infinite amount of solutions for $\dot{X}$ but it is well known [11, pp 193-197] that only $3\cdot N-d$ solutions are needed in order to span the entire set of solutions of the system (12) providing that d is the rank of the $J(\phi)+M$ matrix.

Now taking temporal derivatives in (12), we have:

$$\frac{d\left[J(\phi)\cdot\dot{X}\right]}{dt}=-l\cdot M\cdot\ddot{X}=-F\ , \qquad (13)$$

where we assign a field ϕ to each force F acting over the team. This way is enough to recall the procedure of the Newtonian mechanics [9, pp 3-15] where each force acting over the team could be considered as acting independent of the other forces and we can use the mathematical form of that force obtained without any other interaction except the force under study. The same procedure can be applied considering that we know the j^{th} force and we want to calculate the j^{th} field acting without any other interaction.

It turns out that the time variable is the same for every interaction acting on the team. In the sequel we are considering the fields formed by two parts, the *external* and *internal* fields, as follows:

$$\phi_{ext}+\phi_{\text{int}}=\phi\ , \qquad (14)$$

where ϕ_{ext} is the external field and ϕ_{int} is the internal one.

External fields:

This case include the fields representing the external interactions or interactions between the system and the environment which could be obtained solving from (13) assuming we know the mathematical form of the external forces.

Internal fields:

The case of the internal fields is very different from the external ones because in general we do not know the mathematical form of the internal forces but we know the constraints, which could come in two general forms [13].

It turns out we will interpret the constraints as an equivalent manner to write the internal forces of the team. The next section provides an insight to the constraint equations.

3.2 Constraints

As we mentioned early the restrictions can come in two different ways. For the case of the kinematic (holonomic) restrictions, we have in general [2]:

$$\Delta_k(X)\cdot\dot{X}=0\ , \qquad (15)$$

where Δ_k is a system of k non-linear equations. Determining the null space of the Δ_k matrix we get equations as follows:

$$\dot{X}=\mathrm{H}(X)\ , \qquad (16)$$

where H is a system of k non-linear equations.

For the non-holonomic restrictions we can undertake the problem using the general form as follows [14, pp 28-32]:

$$\Delta'_k(X,\dot{X})=0\ . \qquad (17)$$

Again Δ_k is a system of k non-linear equations.

From (12):

$$\left[J(\phi_{ext}+\phi_{\text{int}})+l\cdot M\right]\cdot\dot{X}=0\Rightarrow\dot{X}=\lambda(t)\cdot H(\phi_{ext}+\phi_{\text{int}})\ ,(18)$$

where $\lambda(t)\in\Re^{nxn}$ is a matrix, which gives the degree of freedom of the equation (18).

Finally incorporating (18) into (16) or (17):

$$\Delta_k(X)\cdot\left[H(\phi_{ext}+\phi_{\text{int}})\right]=0\ \text{(Holonomic case)}\ \ (19.a)$$

$$\Delta'_k(X,\lambda(t)\cdot H(\phi_{ext}+\phi_{\text{int}}))=0\ \text{(Non-Holonomic case)}.(19.a)$$

We notice we still have a coupled partial differential equations system but now without boundary conditions. It is important also to mention that in our case we have just *k-coupled* equations but in the Lagrange's method for the most general non-holonomic case we always have *3N* equations.

Here it is clear we need to calculate first the external fields ϕ_{ext} and after replacing them into (19), we need to determine the internal ones which will be necessary for the calculus of the final motion of the team.

Finally is important that for the pure holonomic case like (19.a) we do not need the matrix $\lambda(t)$ for the calculus of the internal field ϕ_{int}.

3.3 Initial Conditions

The initial conditions are essential in order to get a well-posed problem. Theorem 1 ensures the uniqueness of the solution for our system of partial differential equations (19).

4 Example of application

Let us consider a formation of 3 robots in a planar configuration, where each robot is modelled as a punctual mass. The following holonomic constraints are considered as well:

$$\begin{cases} {x_1}^2 + {y_1}^2 = R^2 \\ {x_2}^2 + {y_2}^2 = R^2 \\ {x_3}^2 + {y_3}^2 = R^2 \end{cases}, \qquad (20)$$

where R is a real number and the constraints are indicating the formation lies in a circle of radius R:

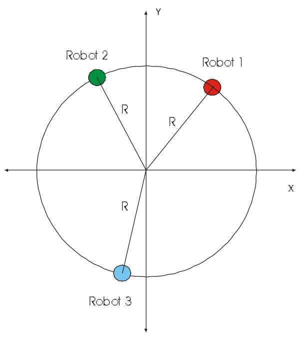

Figure 1: Team in a rigid formation.

Here the constraints regarding the planar configuration of the team are useful to select the adequate coordinates for the problem. In this case we choose $X = [x_1, y_1, \cdots, y_3]$ as Cartesian coordinates, where the vector position of each robot is $[x_i, y_i, z_i]$ with i=1,2,3.

We are considering two external fields: the gravity force and one obstacle field.

For the external fields we can include them in the same mathematical expression but the gravity force has no influence on the team.

Regarding the obstacle field we can consider the following one:

$$F_{obstacle} = \dot{X}_{\phi_{ext}} . \qquad (21)$$

From here we are ready to apply the method depicted in Section 3.1 as follows:

$$F_{obstacle} = \dot{X}_{\phi_{ext}} = M \cdot \ddot{X}_{\phi_{ext}} \Rightarrow X_{\phi_{ext}} = M \cdot \ddot{X}_{\phi_{ext}} + cte , (22)$$

where the initial condition fix the value for the constant in (22) and $X_{\phi ext}$ represents the vector position for the team with only the obstacle force acting on it.

In the next step if we consider the constant in (22) equal to zero and using (13) with $l=1$, we have:

$$-M \cdot \ddot{X}_{\phi_{ext}} = \frac{d\left[J(\phi_{ext}) \cdot \dot{X}_{\phi_{ext}}\right]}{dt} \Rightarrow -X_{\phi_{ext}} = \left[J(\phi_{ext}) \cdot \dot{X}_{\phi_{ext}}\right] . (23)$$

Finally in order to get the external fields in that case we can rewrite (23) in the following way:

$$\begin{cases} -x_{1\phi_{ext}} = \dfrac{\partial^2 \phi_{ext}}{\partial {x_{1\phi_{ext}}}^2} \cdot \dot{x}_{1\phi_{ext}} + \cdots + \dfrac{\partial^2 \phi_{ext}}{\partial y_{3\phi_{ext}} \partial x_{1\phi_{ext}}} \cdot \dot{y}_{3\phi_{ext}} \\ \vdots \\ -y_{3\phi_{ext}} = \dfrac{\partial^2 \phi_{ext}}{\partial x_{1\phi_{ext}} \partial y_{3\phi_{ext}}} \cdot \dot{x}_{1\phi_{ext}} + \cdots + \dfrac{\partial^2 \phi_{ext}}{\partial {y_{3\phi_{ext}}}^2} \cdot \dot{y}_{3\phi_{ext}} \end{cases} . \qquad (24)$$

The equations system (24) is the system to be solved to obtain the external field.

A final step is required to determine the span solution (18) :

$$\dot{X} = \lambda(t) \cdot H(\phi_{ext} + \phi_{int}) , \qquad (25)$$

Rewriting the constraints in (20) in a matrix form we have:

$$\begin{bmatrix} x_1 & y_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 & y_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_3 & y_3 \end{bmatrix} \cdot \dot{X} = A \cdot X = 0 . \qquad (26)$$

Replacing the general solution (25) and the external fields got in (24) into (26) we have:

$$A \cdot (H(\phi_{ext} + \phi_{\text{int}})) = 0 . \qquad (27)$$

Finally solving (27) and incorporing both fields (internal and external) into (18) we get the motion of the team from 6 decoupled ordinary differential equations.

7 Conclusions

In this paper we introduced a novel method to solve the dynamics of robot formations motion. The method presented has several advantages with respect to the traditional Lagrangian or Newtonian models. In the first place we have no boundary conditions on the determination of the internal fields, while in the case of Lagrange's method we have for the most general case a system of partial differential equations with no-integrable constraints; on the other hand the matrix nature of our definition allows to incorporate the constraints in a systematic way through the span solution of a matrix calculus.

On the hard side the task to solve those partial differential equations sometimes has not immediate solutions for the calculus of the fields.

As future work, we plan to use our new model for non-inertial frames and exploit the matrix characteristics for developing stable controllers. On the other hand an appropriate tool for solving (19) will be investigated as well as the systematic way to incorporate the generalized coordinates for choosing the best set of coordinates for every problem.

Acknowledgments

The work of the first author has been supported by the grant [SFRH / BD / 10451 / 2002] from the Portuguese Foundation for Science and Technology.

8 References

[1] Tanner G. Hebert, Savvas G. Loizou and Kostas J. Kyriakopoulos, "Nonholonomic Navigation and Control of Cooperating Mobile Manipulators", *Proc. IEEE Transactions on Robotic and Automation*, 2003

[2] Nilanjan Sarkar, Xiaoping Yun, and Vijay Kumar,"Control of Mechanical Systems with Rolling Constraints: Applications to Dynamic Control of Mobile Robots", The International Journal of Robotics Research, Vol. 13, No. 1, February 1994, pp. 55-69

[3] Milos Zefran, Vijay Kumar and Christopher Croke, On the generation of smooth three-dimensional rigid body motions, *Proc. IEEE Transactions on Robotics and Automation*, 1995.

[4] C. Belta and V. Kumar, "Trajectory design for formations of robots by kinetic energy shaping," *Proc. IEEE Intl. Conf. On Robotics and Automation*, *Washington* D.C., May 2002.

[5] Paulo Tabuada, George J. Pappas, Pedro Lima," Feasible Formations of Multi-Agent Systems" *Proc. American Control Conference, Arlington*, VA, 2001.

[6] Sebastian Thrun, Wolfram Burgard, Dieter Fox, "A probabilistic approach to Concurrents Mapping and Localization for Mobile Robots", Kluwer Academic Publisher, Boston, 1998.

[7] Bruno D. Dama, Pedro U. Lima, Luis M. Custodio, "A modified Potential Fields Method for Robot Navigation Applied to Dribbling in Robotic Soccer", International Robocup Symposium, Fukuoka, 2002.

[8] P. Song and V. Kumar, "A Potential Field Based Approach to Multi-Robot Manipulation," Proc. *IEEE Intl. Conf. On Robotics and Automation* Washington D.C., May 2002.

[9] Keith R. Symon, Mechanics, Addison-Wesley Publishing Company, Inc, 1970.

[10] Robert M. Wald, General Relativity, The University of Chicago Pres, 1984.

[11] Martin Golubitsky and Michael Dellnitz, Linear Algebra and Differential Equations Using Matblab, An International Thomson Company, 1999.

[12] Olver, Peter J., "Applications of Lie groups to differential equations", second edition. Graduate Texts in Mathematics, 107, pp 513. Springer-Verlag, New York, 1993.

[13] Hernán Cendra, Jerrold E. Marsden, Tudor S. Ratiu, "Geometric Mechanics, Lagrangian Reduction, and Nonholonomic Systems", Mathematics Unlimited-2001 and Beyond, pp 28-32, 2001.