52nd IEEE Conference on Decision and Control
December 10-13, 2013. Florence, Italy

Nonlinear Observer for 3D Rigid Body Motion

Sérgio Brás, Maziar Izadi, Carlos Silvestre, Amit Sanyal and Paulo Oliveira

***Abstract*— Observer design for rigid body translational and rotational motion has important applications to unmanned or manned vehicles operating in air, underwater, or in space. An observer design for pose and velocity estimation for three-dimensional rigid body motion, in the framework of geometric mechanics, is presented here. Resorting to convenient defined Lyapunov function, a nonlinear observer on the Special Euclidean Group ($\mathrm{SE}(3)$) is derived. This observer is based on the exponential coordinates, which are used to represent the group of rigid body motions. Exponential convergence of the estimation errors is shown and boundedness of the estimation error under bounded unmodeled torques and forces is established. Since exponential coordinates can describe uniquely almost the entire group of rigid body motions, the resulting observer design is almost globally exponentially convergent. The observer is then applied to the free dynamics of a rigid vehicle. Numerical simulation results are presented to show the performance of this observer, both in the absence and with unmodeled forces and torques.**

I. INTRODUCTION

This paper presents a nonlinear observer that can accurately estimate the configuration and velocity states of a rigid body. Since most unmanned and manned vehicles can be accurately modeled as rigid bodies, the applications of this observer extend to such vehicles operating on air, underwater, and in space. In particular, such vehicles when operated in uncertain or poorly known environments, can be subject to unknown forces and moments. Therefore, robustness of the observer to such unknown disturbances is essential for applications of vehicles in space or underwater exploration. The dynamical coupling between the rotational and translational dynamics, which occurs both due to the natural dynamics, as well as, control forces and torques, is treated in the framework used for our observer design.

The determination of position and attitude is a classical problem in estimation theory. A wide variety of methods have been proposed to address this problem exploiting different techniques and mathematical developments. Some are purely algebraic methods that exploit only position and directions information, others rely on the equations of motion (kinematics and dynamics) in order to integrate inertial measurements and filter the sensor data. Some of the latter, rely on treating the attitude kinematics and dynamics separately from the translational motion, and integrating the measurements of position and attitude with inertial data. For attitude estimation, attitude measurements are obtained indirectly through direction or angle measurements, while angular velocity measurements are obtained from rate gyros, which may or may not be biased. Therefore, many attitude estimation schemes, like those in [1], [2], [3], [4], [5], estimate the attitude and sometimes the rate gyro bias from measurements. The work in [6] proposes a kinematic nonlinear observer which fuses velocity and landmark measurements to provide estimates which converge exponentially fast to the true states. In [7], a gradient-based observer is designed directly on the Special Euclidean Group $\mathrm{SE}(3)$. These geometric estimation schemes used a global attitude description, similar to the observer for mechanical systems on Lie groups presented in [8]. A framework to design pose estimators based on the use of vision sensors is given in [9], [10], [11]. Others solutions have been proposed focusing only on the estimation of attitude or of the position, see for instance [12], [13].

For vehicles with fast changing dynamics, only the kinematic equations might not be sufficient to capture the physical behavior. The advantage of this approach is that, when a suitable model for the forces and moments is available, one can rely on this information to improve the estimates. In [14], the problem of obtaining the angular velocity of a rigid body from orientation and torque measurements only is considered. The work [15] considers two different methods of using a dynamic vehicle model in order to aid pose estimates provided by an inertial navigation system (INS). In [16] a complete model-aided INS for underwater vehicles is presented. A new methodology to exploit the vehicle dynamics based on the extended Kalman filter (EKF) is proposed in [17]. In [18], the attitude of an underwater vehicle is estimated by an observer which considers the rigid body dynamics. Dynamic attitude and angular velocity estimation for uncontrolled rigid bodies using global representation of the equations of motion based on geometric mechanics, is reported in [19], [20]. This estimation scheme was used in [21] for feedback attitude tracking control.

The previous works assume that an accurate model for the vehicle dynamics is available. However, in many situations that might not be the case. For example, spacecraft dynamics may be affected by solar radiation pressure and poorly-known higher-order gravity effects [22]. In [23], an EKF for identification of unmodeled disturbance torques was proposed. The work in [11] also considers pose estimation of rigid bodies affected by disturbance forces and torques.

This article presents an observer design for the configuration and velocities of a rigid vehicle in $\mathrm{SE}(3)$, the Lie group of rigid body translations and rotations. The configuration and velocity error dynamics is shown to be almost globally exponentially stable. The problem of unmodeled disturbances is also addressed. Resorting to Lyapunov analysis, the pro-

S. Brás, P. Oliveira and C. Silvestre are with Laboratory of Robotics and Systems in Engineering and Science, Instituto Superior Técnico, Universidade Técnica de Lisboa, Lisbon 1049-001, Portugal. Prof. Carlos Silvestre is also with Dep. of Electrical and Computer Eng., Faculty of Science and Technology of the University of Macau.

M. Izadi and A. Sanyal are with the Mechanical and Aerospace Engineering department, New Mexico State University, Las Cruces, NM 88011

S. Brás: corresponding author

978-1-4673-5716-6/13/$31.00 ©2013 IEEE

posed observer laws are shown to drive the estimation errors exponentially fast to the origin. Simulations are given that attest the feasibility of the proposed solution.

The remainder of this paper is organized as follows. Section II describes the rigid body equations of motion formulated explicitly on $\mathrm{SE}(3)$ and introduces the state (configuration and velocity) estimation problem. In Section III, a nonlinear observer that estimates pose and velocities is proposed, and its convergence and stability properties are analyzed in Section IV. In Section V, simulation results validating the performance of the proposed observer are presented. Finally, concluding remarks and comments on future work are given in Section VI.

II. Problem Formulation

Consider a body-fixed coordinate frame with origin at the center of mass of a rigid body denoted by $\{B\}$, and an inertially fixed reference frame denoted by $\{I\}$. Let the rotation matrix from $\{B\}$ to $\{I\}$ be given by $\mathbf{R}$ and the coordinates of the origin of $\{B\}$ with respect to $\{I\}$ be denoted by $\mathbf{b}$. The set of rotation matrices, which contains $\mathbf{R}$, is denoted by $\mathrm{SO}(3) = \{\mathbf{R} \in \mathbb{R}^{3\times3} : \mathbf{R}^T\mathbf{R} = \mathbf{I}_3, \det(\mathbf{R}) = 1\}$, where $\mathbf{I}_n$ denotes the $n \times n$ identity matrix. The rigid body kinematics are given by

$$\dot{\mathbf{R}} = \mathbf{R}(\boldsymbol{\omega})^\times, \quad \dot{\mathbf{b}} = \mathbf{R}\mathbf{v},$$

where the linear and angular velocities expressed in the body fixed frame $\{B\}$ are denoted by $\mathbf{v}$ and $\boldsymbol{\omega}$, respectively, and the skew-symmetric operator $(.)^\times : \mathbb{R}^3 \to \mathfrak{so}(3)$ satisfies $(\mathbf{v})^\times\mathbf{w} = \mathbf{v} \times \mathbf{w}$, $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. The linear space $\mathfrak{so}(3)$ is the Lie algebra associated with the Lie group $\mathrm{SO}(3)$ and corresponds to the set of 3×3 skew-symmetric matrices.

Let $\mathcal{G}$ be the rigid body configuration, such that

$$\mathcal{G} = \begin{bmatrix} \mathbf{R} & \mathbf{b} \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix} \in \mathrm{SE}(3),$$

where $\mathbf{0}_{m\times n}$ denotes a $m \times n$ matrix whose all elements are zeros and the *Special Euclidean Group* $\mathrm{SE}(3)$ is characterized by

$$\mathrm{SE}(3) = \left\{ \mathcal{G} \in \mathbb{R}^{4\times4}, \mathcal{G} = \begin{bmatrix} \mathbf{R} & \mathbf{b} \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix} : \begin{array}{l} \mathbf{R} \in \mathrm{SO}(3), \\ \mathbf{b} \in \mathbb{R}^3 \end{array} \right\}.$$

Using this representation, the kinematic equations take the form

$$\dot{\mathcal{G}} = \mathcal{G}\boldsymbol{\xi}^\vee,$$

where $\boldsymbol{\xi} = [\boldsymbol{\omega}^T \ \mathbf{v}^T]^T$ is the vector of body velocities and the vector space isomorphism $(.)^\vee : \mathbb{R}^6 \to \mathfrak{se}(3)$ is given by

$$\boldsymbol{\xi}^\vee = \begin{bmatrix} (\boldsymbol{\omega})^\times & \mathbf{v} \\ \mathbf{0}_{1\times3} & 0 \end{bmatrix} \in \mathfrak{se}(3).$$

The space $\mathfrak{se}(3)$ is the Lie algebra associated with $\mathrm{SE}(3)$ and it consists in a six-dimensional linear space tangent to $\mathrm{SE}(3)$ at the identity element. The rigid body dynamics is given by

$$\mathbf{J}\dot{\boldsymbol{\omega}} = (\mathbf{J}\boldsymbol{\omega})^\times\boldsymbol{\omega} + \boldsymbol{\tau}, \quad m\dot{\mathbf{v}} = (m\mathbf{v})^\times\boldsymbol{\omega} + \boldsymbol{\phi}, \tag{1}$$

where m and $\mathbf{J}$ denote the rigid body scalar mass and inertia matrix, respectively, $\boldsymbol{\phi}$ denotes the force applied to the rigid body and $\boldsymbol{\tau}$ the external torque, both expressed in the body-fixed coordinate frame. The dynamic equations (1) can be expressed in compact form as

$$\mathbb{I}\dot{\boldsymbol{\xi}} = \mathrm{ad}^*_{\boldsymbol{\xi}}\mathbb{I}\boldsymbol{\xi} + \boldsymbol{\varphi},$$

where $\boldsymbol{\varphi} = [\boldsymbol{\tau}^T \ \boldsymbol{\phi}^T]^T$, $\mathrm{ad}^*_{\boldsymbol{\xi}} = (\mathrm{ad}_{\boldsymbol{\xi}})^T$, $\mathbb{I} = \mathrm{diag}(\mathbf{J}, m\mathbf{I}_3)$, and $\mathrm{diag}(\mathbf{A}_1, \ldots, \mathbf{A}_n)$ denotes the block diagonal matrix with the elements $\mathbf{A}_1, \ldots, \mathbf{A}_n$ in the main block diagonal. The operator $\mathrm{ad}_{\boldsymbol{\xi}}$ stands for the linear adjoint representation of the *Lie algebra* $\mathfrak{se}(3)$ associated with the *Lie group* $\mathrm{SE}(3)$ such that

$$\mathrm{ad}_{\boldsymbol{\xi}} = \begin{bmatrix} (\boldsymbol{\omega})^\times & \mathbf{0}_{3\times3} \\ (\mathbf{v})^\times & (\boldsymbol{\omega})^\times \end{bmatrix}.$$

The sensor suite available provides information regarding the configuration, velocities, forces and torques applied to the vehicle. Our aim is to design a dynamic observer which exploits the sensors measurements to estimate the configuration (pose) and the velocities, such that the estimated states converge to their true values in the absence of measurement errors. The use of an observer has clear advantages over the raw measurements as the sensor information is fused with the rigid-body dynamics. The resulting estimates are less noisy and the errors due to sensor bias have smaller magnitude than the raw sensor data. Robustness to bounded measurement errors is obtained consequently, and is shown through numerical simulation results.

III. Observer Synthesis

In this section, we propose an observer for the configuration and velocity. The configuration observer takes the form

$$\dot{\hat{\mathcal{G}}} = \hat{\mathcal{G}}\hat{\boldsymbol{\xi}}^\vee, \quad \hat{\mathcal{G}} = \begin{bmatrix} \hat{\mathbf{R}} & \hat{\mathbf{b}} \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix}, \tag{2}$$

where $\hat{\boldsymbol{\xi}} = [\hat{\boldsymbol{\omega}}^T \ \hat{\mathbf{v}}^T]^T \in \mathbb{R}^6 \simeq \mathfrak{se}(3)$. We define the configuration error as

$$\tilde{\mathcal{G}} = \hat{\mathcal{G}}^{-1}\mathcal{G} = \begin{bmatrix} \tilde{\mathbf{R}} & -\hat{\mathbf{R}}^T\tilde{\mathbf{b}} \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix} \in \mathrm{SE}(3), \tag{3}$$

where $\tilde{\mathbf{R}} = \hat{\mathbf{R}}^T\mathbf{R}$, and $\tilde{\mathbf{b}} = \hat{\mathbf{b}} - \mathbf{b}$. The configuration error can be expressed in exponential coordinates using

$$\tilde{\boldsymbol{\eta}}^\vee = \mathrm{logm}_{\mathrm{SE}(3)}(\tilde{\mathcal{G}}), \tag{4}$$

where $\mathrm{logm}_{\mathrm{SE}(3)}(.) : \mathrm{SE}(3) \to \mathfrak{se}(3)$ denotes the logarithmic map on $\mathrm{SE}(3)$ [24]. The time derivative of configuration error (3) is given by

$$\dot{\tilde{\mathcal{G}}} = \tilde{\mathcal{G}}(\boldsymbol{\xi}^\vee - \mathrm{Ad}_{\tilde{\mathcal{G}}^{-1}}\hat{\boldsymbol{\xi}}^\vee), \tag{5}$$

where the adjoint action of $\mathcal{G} \in \mathrm{SE}(3)$ on $\boldsymbol{\zeta} \in \mathfrak{se}(3)$ is given by $\mathrm{Ad}_{\mathcal{G}}\boldsymbol{\zeta}^\vee = \left(\begin{bmatrix} \mathbf{R} & \mathbf{0}_{3\times3} \\ (\mathbf{b})^\times\mathbf{R} & \mathbf{R} \end{bmatrix} \zeta \right)^\vee$, $\boldsymbol{\zeta} \in \mathbb{R}^6$, $\mathcal{G} \in \mathrm{SE}(3)$. In this work, we assume that full state measurement is provided by the sensor suite on board the vehicle. Thus, we pose the following assumption.

Assumption 1: The available sensor suite provides measurements about the configuration, velocity, forces and torques applied to the vehicle.

Note that, even with full state measurements, the e existence of an observer is valuable for any navigation and control system as, like the EKF, it can mitigate the effects of sensor uncertainties such as noise and bias.

Let us now define the following quantities

$$\breve{\boldsymbol{\xi}}^\vee = \mathrm{Ad}_{\tilde{\mathcal{G}}^{-1}} \hat{\boldsymbol{\xi}}^\vee, \qquad \tilde{\boldsymbol{\xi}} = \boldsymbol{\xi} - \breve{\boldsymbol{\xi}}, \tag{6}$$

Updating (5), it results in

$$\dot{\tilde{\mathcal{G}}} = \tilde{\mathcal{G}} \tilde{\boldsymbol{\xi}}^\vee. \tag{7}$$

We express the exponential coordinate vector $\tilde{\boldsymbol{\eta}}$ for the pose estimate error as

$$\tilde{\boldsymbol{\eta}} = \begin{bmatrix} \tilde{\boldsymbol{\Theta}} \\ \tilde{\boldsymbol{\beta}} \end{bmatrix} \in \mathbb{R}^6 \simeq \mathfrak{se}(3),$$

where $\tilde{\boldsymbol{\Theta}} \in \mathbb{R}^3 \simeq \mathfrak{so}(3)$ is the exponential coordinate vector (principal rotation vector) for the attitude estimation error and $\tilde{\boldsymbol{\beta}} \in \mathbb{R}^3$ is the exponential coordinate vector for the position estimate error. The time derivative of the exponential coordinates of the configuration error are derived resorting to (7) and [24]

$$\dot{\tilde{\boldsymbol{\eta}}} = G(\tilde{\boldsymbol{\eta}})\tilde{\boldsymbol{\xi}}, \quad G(\tilde{\boldsymbol{\eta}}) = \begin{bmatrix} A(\tilde{\Theta}) & \mathbf{0}_{3\times 3} \\ T(\tilde{\boldsymbol{\Theta}}, \tilde{\boldsymbol{\beta}}) & A(\tilde{\boldsymbol{\Theta}}) \end{bmatrix}, \tag{8}$$

where

$$\begin{aligned} A(\tilde{\boldsymbol{\Theta}}) &= \mathbf{I}_3 + \frac{1}{2}\tilde{\boldsymbol{\Theta}}^\times + \left(\frac{1}{\theta^2} - \frac{1+\cos\theta}{2\theta\sin\theta}\right)\left(\tilde{\boldsymbol{\Theta}}^\times\right)^2, \\ S(\tilde{\boldsymbol{\Theta}}) &= \mathbf{I}_3 + \frac{1-\cos\theta}{\theta^2}\tilde{\boldsymbol{\Theta}}^\times + \frac{\theta - \sin\theta}{\theta^3}\left(\tilde{\boldsymbol{\Theta}}^\times\right)^2, \text{ and} \\ T(\tilde{\boldsymbol{\Theta}}, \tilde{\boldsymbol{\beta}}) &= \frac{1}{2}\left(S(\tilde{\boldsymbol{\Theta}})\tilde{\boldsymbol{\beta}}\right)^\times A(\tilde{\boldsymbol{\Theta}}) \\ &\quad + \left(\frac{1}{\theta^2} - \frac{1+\cos\theta}{2\theta\sin\theta}\right)\left[\tilde{\boldsymbol{\Theta}}\tilde{\boldsymbol{\beta}}^T + (\tilde{\boldsymbol{\Theta}}^T\tilde{\boldsymbol{\beta}})A(\tilde{\boldsymbol{\Theta}})\right] \\ &\quad - \frac{(1+\cos\theta)(\theta - \sin\theta)}{2\theta\sin^2\theta} S(\tilde{\boldsymbol{\Theta}})\tilde{\boldsymbol{\beta}}\tilde{\boldsymbol{\Theta}}^T \\ &\quad + \left(\frac{(1+\cos\theta)(\theta + \sin\theta)}{2\theta^3\sin^2\theta} - \frac{2}{\theta^4}\right)\tilde{\boldsymbol{\Theta}}^T\tilde{\boldsymbol{\beta}}\tilde{\boldsymbol{\Theta}}\tilde{\boldsymbol{\Theta}}^T, \end{aligned}$$

where $\theta = \|\tilde{\boldsymbol{\Theta}}\|$. The exponential coordinate vector $\tilde{\boldsymbol{\Theta}}$ for the rotational motion and its time derivative are obtained from Rodrigues' formula

$$R(\tilde{\boldsymbol{\Theta}}) = \mathbf{I}_3 + \frac{\sin\theta}{\theta}\tilde{\boldsymbol{\Theta}}^\times + \frac{1-\cos\theta}{\theta^2}((\tilde{\boldsymbol{\Theta}})^\times)^2,$$

which is a well-known formula for the rotation matrix in terms of the exponential coordinates on $\mathrm{SO}(3)$, the Lie group of special orthogonal matrices. In the context of (8), the matrix $R(\tilde{\Theta}) = \tilde{\mathbf{R}}$, i.e., the attitude estimate error on $\mathrm{SO}(3)$. We consider next a result that is important in obtaining the observer described later in this section.

Lemma 1: The matrix $G(\tilde{\boldsymbol{\eta}})$, which occurs in the kinematics (8) *for the exponential coordinates on* $\mathrm{SE}(3)$*, satisfies the relation $G(\tilde{\boldsymbol{\eta}})\tilde{\boldsymbol{\eta}} = \tilde{\boldsymbol{\eta}}$.*

Proof: In [24], an expansion for $G(\tilde{\boldsymbol{\eta}})$ is given in terms of matrix powers of $\mathrm{ad}_{\tilde{\boldsymbol{\eta}}}$, from which the above result can be easily concluded given that $\mathrm{ad}_{\boldsymbol{\chi}}\boldsymbol{\chi} = 0$, $\boldsymbol{\chi} \in \mathfrak{se}(3)$. ■

Remark 1: The exponential coordinate $\tilde{\boldsymbol{\Theta}}$ for SO(3) cannot be uniquely obtained when $\theta = \|\tilde{\boldsymbol{\Theta}}\| = \pi$ radians, since $\tilde{\boldsymbol{\Theta}}$ and $-\tilde{\boldsymbol{\Theta}}$ give the same rotation matrix in this case, according to Rodrigues' formula. In this case, the matrix $G(\tilde{\boldsymbol{\eta}})$ also becomes singular.

Consider the following Lyapunov function candidate

$$V = \frac{1}{2}\tilde{\boldsymbol{\eta}}^T\mathbf{K}\tilde{\boldsymbol{\eta}} + \frac{1}{2}(k_1\tilde{\boldsymbol{\eta}} + \tilde{\boldsymbol{\xi}})^T\mathbf{K}\mathbb{I}(k_1\tilde{\boldsymbol{\eta}} + \tilde{\boldsymbol{\xi}}), \tag{9}$$

where $\mathbf{K} = \mathrm{diag}(\mathbf{I}_3, k_2\mathbf{I}_3)$, $k_1, k_2 > 0$, which motivates the development of the velocity observer. Letting $\mathbf{u} = k_1\tilde{\boldsymbol{\eta}} + \tilde{\boldsymbol{\xi}}$ and taking the time derivative produces

$$\begin{aligned} \dot{V} = &- k_1\tilde{\boldsymbol{\eta}}^T\mathbf{K}\tilde{\boldsymbol{\eta}} + \mathbf{u}^T\mathbf{K}(G^T(\mathbf{K}\tilde{\boldsymbol{\eta}})\tilde{\boldsymbol{\eta}} + k_1\mathbb{I}G(\tilde{\boldsymbol{\eta}})\tilde{\boldsymbol{\xi}} \\ &+ \mathrm{ad}^*_{\boldsymbol{\xi}}\mathbb{I}\boldsymbol{\xi} + \boldsymbol{\varphi} - \mathbb{I}\dot{\breve{\boldsymbol{\xi}}}), \end{aligned}$$

where it is exploited the equality $\mathbf{K}^{-1}G^T(\tilde{\boldsymbol{\eta}})\mathbf{K} = G^T(\mathbf{K}\tilde{\boldsymbol{\eta}})$. Let

$$\mathbb{I}\dot{\breve{\boldsymbol{\xi}}} = \mathrm{ad}^*_{-\mathbf{K}(k_1\tilde{\boldsymbol{\eta}} - \tilde{\boldsymbol{\xi}})}\mathbb{I}\boldsymbol{\xi} + \boldsymbol{\varphi} + k_1\mathbb{I}G(\tilde{\boldsymbol{\eta}})\tilde{\boldsymbol{\xi}} + G^T(\mathbf{K}\tilde{\boldsymbol{\eta}})\tilde{\boldsymbol{\eta}} + k_3\mathbf{u}, \tag{10}$$

where $k_3 > 0$. Then, resorting to some algebraic manipulations, the time derivative of (9) takes the negative definite form

$$\dot{V} = -k_1\tilde{\boldsymbol{\eta}}^T\mathbf{K}\tilde{\boldsymbol{\eta}} - k_3(k_1\tilde{\boldsymbol{\eta}} + \tilde{\boldsymbol{\xi}})^T\mathbf{K}(k_1\tilde{\boldsymbol{\eta}} + \tilde{\boldsymbol{\xi}}).$$

Thus, the point $(\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\xi}}) = (\mathbf{0}, \mathbf{0})$ is asymptotically stable in sense of Lyapunov [25]. Topological limitations precludes global asymptotic stability of the origin [26]. In fact, if $\theta = \pi$, the exponential coordinates of the configuration error $\tilde{\boldsymbol{\eta}}$ cannot be computed without ambiguity from sensors. The next proposition provides sufficient conditions ensuring that for all $t > t_0$, $\theta(t) < \pi$.

Lemma 2: Let $\underline{\sigma}_{\mathbf{J}} = \sigma_{\min}(\mathbf{J})$ and $\bar{\sigma}_{\mathbf{J}} = \sigma_{\max}(\mathbf{J})$ denote the minimum and maximum singular value of $\mathbf{J}$, respectively. For any initial condition such that

$$\|\tilde{\boldsymbol{\Theta}}_0\|^2 + c_1\|\tilde{\mathbf{b}}_0\|^2 + c_2(\bar{\sigma}_{\mathbf{J}}\|\tilde{\boldsymbol{\Theta}}_0\| + k_2\mu m\|\tilde{\mathbf{b}}_0\|) + c_3 < \pi^2,$$

where $\mu = \sqrt{1 + \pi^2/2}$, $c_1 = \frac{k_2\mu(1+k_1^2 m)}{1+k_1^2\underline{\sigma}_{\mathbf{J}}}$, $c_2 = \frac{2k_1\|\tilde{\boldsymbol{\xi}}_0\|}{1+k_1^2\underline{\sigma}_{\mathbf{J}}}$, $c_3 = \frac{\tilde{\boldsymbol{\xi}}_0^T\mathbb{I}\tilde{\boldsymbol{\xi}}_0}{1+k_1^2\underline{\sigma}_{\mathbf{J}}}$, $\tilde{\boldsymbol{\Theta}}_0 = \tilde{\boldsymbol{\Theta}}(t_0)$, $\|\tilde{\mathbf{b}}_0\| = \tilde{\mathbf{b}}(t_0)$, and $\|\tilde{\boldsymbol{\xi}}_0\| = \tilde{\boldsymbol{\xi}}(t_0)$, there is $\bar{\theta} < \pi$ such that the exponential representation of the attitude error satisfies $\|\tilde{\boldsymbol{\Theta}}(t)\| \leq \bar{\theta}$ for all $t > t_0$. Moreover, there is a one-to-one mapping between the Lie group $\tilde{\mathcal{G}} \in \mathrm{SE}(3)$ and its Lie algebra (exponential) representation along all the trajectories of the system.

Proof: The exponential coordinates of the configuration error are related to $\tilde{\mathbf{R}}$ and $\tilde{\mathbf{b}}$ by [24]

$$\begin{aligned} \tilde{\boldsymbol{\Theta}} &= \frac{\rho}{\sin\rho}(\tilde{\mathbf{R}} - \tilde{\mathbf{R}}^T), \ \cos\rho = \frac{\mathrm{tr}(\tilde{\mathbf{R}}) - 1}{2}, |\rho| < \pi, \\ \tilde{\boldsymbol{\beta}} &= S^{-1}(\tilde{\boldsymbol{\Theta}})\tilde{\mathbf{b}}, \end{aligned} \tag{11}$$

where $\mathrm{tr}(\tilde{\mathbf{R}}) \neq -1$, $\mathrm{tr}(.)$ denotes the trace operator, and

$$S^{-1}(\tilde{\boldsymbol{\Theta}}) = \mathbf{I}_3 - \frac{1}{2}(\tilde{\boldsymbol{\Theta}})^\times + \frac{2\sin\theta - \theta(1-\cos\theta)}{2\theta^2\sin\theta}((\tilde{\boldsymbol{\Theta}})^\times)^2.$$

From the relation between matrix norms [27] we have

$$\|S^{-1}(\tilde{\boldsymbol{\Theta}})\tilde{\mathbf{b}}\| \leq \|S^{-1}(\tilde{\boldsymbol{\Theta}})\|_2\|\tilde{\mathbf{b}}\| \leq \|S^{-1}(\tilde{\boldsymbol{\Theta}})\|_F\|\tilde{\mathbf{b}}\|,$$

where $\|.\|_2$ and $\|.\|_F$ denote the Euclidean and Forbenius norms of matrices, respectively. Through some algebraic manipulations one obtains $\|S^{-1}(\tilde{\mathbf{\Theta}})\|_F \leq \mu$, where $\mu = \sqrt{1+\pi^2/2}$. Hence, $\|\tilde{\boldsymbol{\eta}}\|^2 \leq \|\tilde{\mathbf{\Theta}}\|^2 + \mu^2\|\tilde{\mathbf{b}}\|^2$. Consider the level set C_V of the Lyapunov function (9) defined as $C_V = \left\{\mathbf{x} : V(\mathbf{x},t) < \frac{\pi^2}{2}\left(1+k_1^2\underline{\sigma}_{\mathbf{J}}\right)\right\}$, where $\mathbf{x} = (\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\xi}})$. The condition (11) guarantees that $\mathbf{x}(t=t_0) \in C_V$. Since C_V is a positive invariant set we have $\mathbf{x}(t) \in C_V$. We conclude the proof by noting that $\mathbf{x}(t) \in C_V \Rightarrow \|\tilde{\mathbf{\Theta}}(t)\| \leq \pi$ for all $t \geq t_0$. ■

Note that, for all initial conditions such that $\|\tilde{\mathbf{\Theta}}_0\| = \pi - \epsilon$, $\epsilon > 0$, the tuning parameters of the proposed observer, k_1 and k_2, can always be selected such that (11) is satisfied. To summarize, the observer design is given by equations (2) and (10) . This observer directly integrates and propagates the configuration (pose) on SE(3) and the velocities on $\mathfrak{se}(3)$, using the exponential coordinates for the pose estimate error, given by $\tilde{\boldsymbol{\eta}}$ in equation (4).

IV. Observer Properties

This section evidences important characteristics of the observer. Namely, it is shown that the observer is almost globally exponentially stable and that, when there is bounded perturbations in the forces and torques measurements, the estimation errors are uniformly ultimately bounded. Explicit convergence bounds for the estimation error are also presented.

Lemma 3: The Lyapunov function (9) *and the corresponding time derivative satisfy the following bounds*

$$\alpha_1\|\mathbf{x}(t)\|^2 \leq V(\mathbf{x},t) \leq \alpha_2\|\mathbf{x}(t)\|^2,$$
$$\dot{V}(\mathbf{x},t) \leq -\alpha_3\|\mathbf{x}(t)\|^2,$$

where $\alpha_1 = \sigma_{\min}(\mathbf{P})$*,* $\alpha_2 = \sigma_{\max}(\mathbf{P})$*,* $\alpha_3 = \sigma_{\min}(\mathbf{Q})$*, with*

$$\mathbf{P} = \frac{1}{2}\begin{bmatrix}\mathbf{K}(\mathbf{I}_6 + k_1^2\mathbb{I}) & k_1\mathbf{K}\mathbb{I} \\ k_1\mathbf{K}\mathbb{I} & \mathbf{K}\mathbb{I}\end{bmatrix},$$
$$\mathbf{Q} = \begin{bmatrix}(k_1+k_1^2k_3)\mathbf{K} & k_1k_3\mathbf{K} \\ k_1k_3\mathbf{K} & k_3\mathbf{K}\end{bmatrix}.$$

Moreover, we have that

$$\alpha_1 \geq \frac{1}{2}\min\{1,k_2\}(-k_1\underline{\sigma}_{\mathbb{I}} + \min\{1+k_1^2\underline{\sigma}_{\mathbb{I}}, \underline{\sigma}_{\mathbb{I}}\})$$
$$\alpha_2 \leq \frac{1}{2}\max\{1,k_2\}(k_1\bar{\sigma}_{\mathbb{I}} + \max\{1+k_1^2\bar{\sigma}_{\mathbb{I}}, \bar{\sigma}_{\mathbb{I}}\})$$
$$\alpha_3 \geq \min\{1,k_2\}(-k_1k_3 + \min\{k_1+k_1^2k_3, k_3\}),$$

where $\underline{\sigma}_{\mathbb{I}} = \sigma_{\min}(\mathbb{I})$ *and* $\bar{\sigma}_{\mathbb{I}} = \sigma_{\max}(\mathbb{I})$*.*

Proof: Let $\underline{\rho} = \min\{1,k_2\}$ and $\bar{\rho} = \max\{1,k_2\}$. To compute α_1 and α_2 start by noting that

$$V(\mathbf{x},t) = \mathbf{x}^T\mathbf{P}\mathbf{x},$$

Following [28, Theorem 6], we have that

$$\sigma_{\min}\left(\mathrm{diag}(\mathbf{A},\mathbf{C})\right) - \sigma_{\min}(\mathbf{B}) \leq \sigma_{\min}\left(\begin{bmatrix}\mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{C}\end{bmatrix}\right), \quad (12)$$

for any square matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. Exploiting this result, we conclude that

$$2\sigma_{\min}(\mathbf{P}) \geq \sigma_{\min}\left(\mathrm{diag}(\mathbf{K}(\mathbf{I}_6+k_1^2\mathbb{I}), \mathbf{K}\mathbb{I})\right) - \sigma_{\min}(k_1\mathbf{K}\mathbb{I})$$
$$\geq k_1\underline{\rho}(-k_1\underline{\sigma}_{\mathbb{I}} + \min\{1+k_1^2\underline{\sigma}_{\mathbb{I}}, \underline{\sigma}_{\mathbb{I}}\}).$$

From [28, Theorem 6], we have that

$$\sigma_{\max}\left(\begin{bmatrix}\mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{C}\end{bmatrix}\right) \leq \sigma_{\max}\left(\mathrm{diag}(\mathbf{A},\mathbf{C})\right) + \sigma_{\max}(\mathbf{B}).$$

Thus, $\sigma_{\max}(\mathbf{P}) \leq \frac{1}{2}\|\mathbf{x}\|^2\bar{\rho}(k_1\bar{\sigma}_{\mathbb{I}} + \max\{1+k_1^2\bar{\sigma}_{\mathbb{I}}, \bar{\sigma}_{\mathbb{I}}\})$.

Finally, α_3 is obtained resorting to the same reasoning as in α_1 and α_2. Note that

$$-\dot{V}(\mathbf{x},t) = \mathbf{x}^T\mathbf{Q}\mathbf{x},$$

Thus, $-\dot{V}(\mathbf{x},t) \geq \sigma_{\min}(\mathbf{Q})$ and using (12), we conclude that $\sigma_{\min}(\mathbf{Q}) \geq \sigma_{\min}\left(\mathrm{diag}((k_1+k_1^2k_3)\mathbf{K}, k_3\mathbf{K}\mathbb{I})\right) - \sigma_{\min}(k_1k_3\mathbf{K}) \geq \underline{\rho}(-k_1k_3 + \min\{k_1+k_1^2k_3, k_3\})$. ■

A. Almost global exponential stability

The following theorem characterizes the stability of the estimation errors and provides explicit convergence bounds.

Theorem 1: Under Assumption 1, let the configuration and velocities observer be given by (2)*,* (10) *and* (6)*, and* $k_1, k_2, k_3 > 0$ *be such that the the conditions of Lemma 2 are satisfied. Then, the estimation errors* $\mathbf{x} = (\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\xi}})$ *are almost globally exponentially stable with convergence rate upper bounded by*

$$\|\mathbf{x}(t)\| \leq \kappa e^{-\gamma(t-t_0)}\|\mathbf{x}(t_0)\|,$$

where $\kappa = \sqrt{\alpha_2/\alpha_1}$ *and* $\gamma = \alpha_3/(2\alpha_2)$*.*

Proof: Lemma 3 shows that V satisfies $\dot{V} \leq -\frac{\alpha_3}{\alpha_2}V$. Thus, we conclude that

$$V(\mathbf{x},t) \leq e^{-\frac{\alpha_3}{\alpha_2}(t-t_0)}V(\mathbf{x}(t_0),t).$$

Again from Lemma 3, we have that $V(\mathbf{x}(t_0),t) \leq \alpha_2\|\mathbf{x}(t_0)\|^2$, $V(\mathbf{x},t) \geq \alpha_1\|\mathbf{x}(t)\|^2$. Then, $\alpha_1\|\mathbf{x}(t)\|^2 \leq e^{-\frac{\alpha_3}{\alpha_2}(t-t_0)}\alpha_2\|\mathbf{x}(t_0)\|^2$, and consequently,

$$\|\mathbf{x}(t)\| \leq \sqrt{\alpha_2/\alpha_1}e^{-\frac{\alpha_3}{2\alpha_2}(t-t_0)}\|\mathbf{x}(t_0)\|$$

. ■

Note that satisfaction of the conditions of Lemma 2 ensure that unwinding does not occur for the attitude estimate obtained with this observer design. In other words, the norm of the attitude estimate error expressed in exponential coordinates always remains bounded above by π radians.

B. Unmodeled torques and forces

In real mission scenarios, the rigid body can be affected by external time-varying unmodeled disturbances. The following proposition shows that, for bounded unmodeled forces and torques, the configuration and velocity estimates are uniformly ultimately bounded with ultimate bound function of the observer gains.

Assumption 2: The forces and torques applied to the rigid body are composed of modeled and unmodeled components, such that,

$$\boldsymbol{\varphi}(t) = \boldsymbol{\varphi}_r(t) + \boldsymbol{\varphi}_d(t),$$

where $\boldsymbol{\varphi}_r \in \mathbb{R}^6$ and $\boldsymbol{\varphi}_d \in \mathbb{R}^6$ denote the modeled and unmodeled torques and forces, respectively. Moreover, $\boldsymbol{\varphi}_d(t)$ is uniformly bounded, i.e., there exists $\bar{\varphi}_d > 0$ such that, for all $t > t_0$, $\|\boldsymbol{\varphi}_d(t)\| \le \bar{\varphi}_d$.

In the presence of unmodeled forces and torques, Lemma 2 is no longer valid and an additional condition is needed to guarantee that the exponential coordinates of $\tilde{\mathcal{G}}$ can be computed uniquely for all $t > t_0$.

Lemma 4: Under Assumption 2, let the velocities observer be given by

$$\mathbb{I}\dot{\check{\boldsymbol{\xi}}} = \mathrm{ad}^*_{-\mathbf{K}(k_1\tilde{\boldsymbol{\eta}}-\check{\boldsymbol{\xi}})}\mathbb{I}\boldsymbol{\xi} + \boldsymbol{\varphi}_r + k_1\mathbb{I}G(\tilde{\boldsymbol{\eta}})\tilde{\boldsymbol{\xi}} + G^T(\mathbf{K}\tilde{\boldsymbol{\eta}})\tilde{\boldsymbol{\eta}} + k_3\mathbf{u}. \tag{14}$$

Then, if the conditions (11) *and*

$$\min\{k_1^2, k_3^2/\bar{\sigma}_{\mathbb{I}}^2\}(1 + k_1^2\underline{\sigma}_{\mathbf{J}}) > \frac{\bar{\rho}^2}{\pi^2\underline{\rho}\underline{\sigma}_{\mathbb{I}}}\bar{\varphi}_d^2.$$

are satisfied, there is $\bar{\theta} < \pi$ such that $\|\tilde{\boldsymbol{\Theta}}(t)\| \le \bar{\theta}$ for all $t \ge t_0$ and there is a one-to-one mapping between $\tilde{\mathcal{G}} \in$ SE(3) *and its exponential representation along all the system trajectories.*

Proof: From Lemma 2, we have that in order to the attitude error satisfy $\|\tilde{\boldsymbol{\Theta}}(t)\| \le \bar{\theta}$, $\bar{\theta} < \pi$ and to exist a one-to-one mapping between $\tilde{\mathcal{G}} \in \mathrm{SE}(3)$ and its exponential representation for all $t > t_0$, the following condition is necessary

$$V(\mathbf{x}(t)) < \frac{\pi^2}{2}(1 + k_1^2\underline{\sigma}_{\mathbf{J}}). \tag{15}$$

This inequality defines a level set with boundary given by

$$\begin{aligned}\Omega = \{\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\xi}} \in \mathbb{R}^6 : \tilde{\boldsymbol{\eta}}^T\mathbf{K}\tilde{\boldsymbol{\eta}} &+ (k_1\tilde{\boldsymbol{\eta}} + \tilde{\boldsymbol{\xi}})^T\mathbf{K}\mathbb{I}(k_1\tilde{\boldsymbol{\eta}} + \tilde{\boldsymbol{\xi}}) \\ &= \pi^2(1 + k_1^2\underline{\sigma}_{\mathbf{J}})\}.\end{aligned}$$

Our goal is to provide sufficient conditions such that $\dot{V}(\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\xi}}) < 0$ for any $(\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\xi}}) \in \Omega$, which guarantees that (15) holds.

The time derivative of the Lyapunov function, V, taking into account unmodeled forces and torques, satisfies

$$\begin{aligned}\dot{V} \le &-\min\{k_1, k_3/\bar{\sigma}_{\mathbb{I}}\}(\tilde{\boldsymbol{\eta}}^T\mathbf{K}\tilde{\boldsymbol{\eta}} + \mathbf{u}^T\mathbf{K}\mathbb{I}\mathbf{u}) \\ &+ \sqrt{\tilde{\boldsymbol{\eta}}^T\mathbf{K}\tilde{\boldsymbol{\eta}} + \mathbf{u}^T\mathbf{K}\mathbb{I}\mathbf{u}}\sqrt{\bar{\rho}^2/(\underline{\rho}\underline{\sigma}_{\mathbb{I}})}\bar{\varphi}_d.\end{aligned}$$

Evaluating $\dot{V}$ at Ω, it can be concluded that

$$\begin{aligned}\dot{V}((\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\xi}}) \in \Omega) \le &-\min\{k_1, k_3/\bar{\sigma}_{\mathbb{I}}\}\pi^2(1 + k_1^2\underline{\sigma}_{\mathbf{J}}) \\ &+ \sqrt{\pi^2(1 + k_1^2\underline{\sigma}_{\mathbf{J}})}\sqrt{\bar{\rho}^2/(\underline{\rho}\underline{\sigma}_{\mathbb{I}})}\bar{\varphi}_d.\end{aligned} \tag{16}$$

Then, from (16) we conclude that $\dot{V}((\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\xi}}) \in \Omega) < 0$ holds for $\min\{k_1^2, k_3^2/\bar{\sigma}_{\mathbb{I}}^2\}(1 + k_1^2\underline{\sigma}_{\mathbf{J}}) > \frac{\bar{\rho}^2}{\pi^2\underline{\rho}\underline{\sigma}_{\mathbb{I}}}\bar{\varphi}_d^2$. ■

If there are unmodeled forces and torques, the estimation errors are no longer asymptotically stable. However, it can be shown that, with $\|\boldsymbol{\varphi}_d\| \le \bar{\varphi}_d$, for a sufficiently small initial estimation error, the estimation errors are uniformly bounded with ultimate bound proportional to $\bar{\varphi}_d$.

Theorem 2: Under the conditions of Lemma 4, the estimation errors are uniformly ultimately bounded, with ultimate bound given by $\sqrt{\frac{\alpha_2}{\alpha_1}\frac{\alpha_4}{\alpha_3}\bar{\boldsymbol{\varphi}}_d}$, where $\alpha_4 =$ $2\max\{k_1, k_2, k_1k_2\}$.

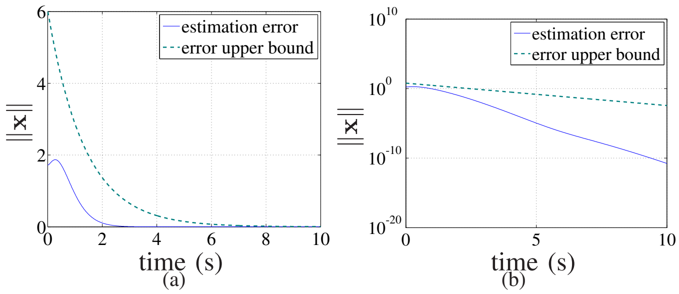

Fig. 1. Estimation error in linear (a) and logarithmic scales (b).

Proof: Under Assumption 2 and (14), we have that

$$\begin{aligned}\dot{V} &< -\alpha_3\|\mathbf{x}\|^2 + \|\mathbf{x}\|2\bar{\boldsymbol{\varphi}}_d\max\{k_1, k_2, k_1k_2\} \\ &\le -\|\mathbf{x}\|(\alpha_3\|\mathbf{x}\| - \bar{\boldsymbol{\varphi}}_d\alpha_4),\end{aligned}$$

where $\alpha_4 = 2\max\{k_1, k_2, k_1k_2\}$. Thus, $\dot{V} < 0$ for $\|\mathbf{x}\| > \frac{\alpha_4}{\alpha_3}\bar{\boldsymbol{\varphi}}_d$ and the set $I = \left\{\mathbf{x} : V(\mathbf{x}, t) < \left(\frac{\alpha_4}{\alpha_3}\bar{\boldsymbol{\varphi}}_d\right)^2\right\}$ is positive invariant. Finally, from the results in Lemma 3, we conclude that

$$\|\mathbf{x}(t)\| \in I \Rightarrow \|\mathbf{x}(t)\| \le \sqrt{\frac{\alpha_2}{\alpha_1}\frac{\alpha_4}{\alpha_3}\bar{\boldsymbol{\varphi}}_d}. \tag{17}$$

■

V. Simulations

In this section, simulation results for the proposed nonlinear observer are presented. The performance of the solution is studied for a typical trajectory. This trajectory is generated by torques and forces with an oscillatory profile with frequency of $\frac{1}{2\pi}$Hz and amplitude of $\boldsymbol{\tau} = 0.1[1.5\ {-2}\ 1]^T$ rad/s^2 and $\boldsymbol{\phi} = [-3\ 2\ 1]^T$ m/s^2. The inertia of the rigid body is given by $\mathbf{J} = \mathrm{diag}(1.1, 1, 0.9)$ kg.m^2, $m = 2$ kg.

We performed simulations under two different conditions, perfect sensor measurements and with unmodeled torques and forces. The exponential convergence of the estimation errors is studied first. The observer parameters are given by $k_1 = 1$, $k_2 = 1$, $k_3 = 4$, $\tilde{\boldsymbol{\Theta}}(t_0) = [-0.4\ {-0.2}\ {-0.1}]^T$ rad, $\tilde{\boldsymbol{\beta}} = [-1.073\ \ {-0.349}\ \ 0.488]^T$ m, $\tilde{\boldsymbol{\omega}}(t_0) = 10^{-3}[7\ 4\ 10]^T$ rad/s, $\tilde{\mathbf{v}}(t_0) = 10^{-3}[10\ 0\ {-5}]^T$ m/s. Note that these parameters satisfy the condition (11).

The configuration and velocity estimation results are depicted in Fig. 1 in both linear and logarithmic scales. The exponential upper bound derived in Section IV-A is also shown. The robustness properties of the observer are illustrated by considering the presence of unmodeled torques and forces with an oscillatory profile with frequency 1 Hz and amplitude 0.01 rad/s^2 and 0.01 m/s^2 in each axis of the torques and forces, respectively. The norm of the unmodeled torques and forces is depicted in Fig. 2(a). The estimation error is shown in logarithmic scale in Fig. 2(b) where, the estimation error converges in finite time to the region defined by (17).

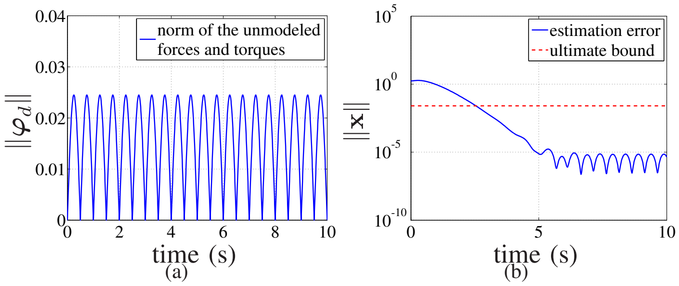

Fig. 2. Norm of the unmodeled forces and torques, $\varphi_d(t)$ (a), and estimation error in the presence of unmodeled forces and torques (b).

VI. Conclusions

A nonlinear observer for arbitrary rigid body motion with full state measurements was devised. This observer was obtained and expressed in terms of the exponential coordinates on the group of rigid body motions in three-dimensional Euclidean space. This observer was shown to be exponentially stable whenever the exponential coordinates are defined, which includes all attitude estimate errors except those corresponding to a principal rotation angle of 180° or π radians. Therefore, the convergence of estimates given by this observer was almost global over the state space of rigid body motion. Boundedness of the estimation error under bounded unmodeled torques and forces was established. Numerical simulation results confirmed the convergence and stability properties of this observer.

Future work will address the stability analysis for the cases with sensor noise in all measurements and in the absence of some measurements.

Acknowledgments

The work of S. Brás, P. Oliveira and C. Silvestre was supported in part by Fundação para a Ciência e Tecnologia (FCT) under project PEst-OE/EEI/LA0009/2011, projects PTDC/EEA-CRO/102857/2008 SCARVE, and PTDC/HIS-ARQ/103227/2008 AMMAIA. The work of S. Brás was supported by the PhD grant SFRH/BD/47456/2008 from FCT. The work of M. Izadi and A. Sanyal was supported in part by NSF under grant CMMI 1131643.

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