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LMI-based $\mathcal{H}_2$ adaptive filtering for 3D positioning and tracking systems

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***Abstract*— In this work, a cascade of two estimators is proposed as the solution for a joint parameter and state estimation problem associated with a target maneuvering in the three-dimensional space. A model for the target that depends on its angular speed is considered and only the target position is measured. A parameter identifier is used to obtain estimates of the target angular speed, which are then fed into an adaptive filter that estimates the position, linear velocity, and linear acceleration of the target. The synthesis of the parameter identifier resorts to Lyapunov techniques and the synthesis of the adaptive filter is tackled using Linear Matrix Inequalities (LMIs) and $\mathcal{H}_2$ optimization strategies. Under persistence of excitation conditions, the error in the angular speed identification and the error in the target state estimates provided by the $\mathcal{H}_2$ adaptive filter are: i) proved to converge exponentially fast to zero in the deterministic setup, i.e., in the absence of noise, and ii) proved to be bounded when bounded stochastic disturbances are considered and there is an upper bound on the target linear velocity and angular speed. To assess the proposed methods, simulations showing that the aforementioned stability and convergence properties hold, even when the estimates provided by an Extended Kalman Filter diverge, are presented.**

I. INTRODUCTION

The problem of 3D target positioning and tracking has been widely studied over the last decades, mainly due to the great impact that the availability of reliable estimates for the position of a target has in the performance of many robotic applications. Some examples of such applications appear, for instance, in the contexts of security and surveillance, trajectory determination, human-computer interaction, and air traffic control, see [1], [2], [3], and [4].

Positioning and tracking consist in using measurements provided by one or more sensors, at fixed locations or at moving platforms, to estimate the state of a moving object, which is usually composed of its position, velocity, and sometimes acceleration. To estimate such quantities, a dynamical model for the maneuvering target is usually considered, see the comprehensive survey in [5]. Typical models commonly depend on the target angular velocity, or on its magnitude, the target angular speed. However, most of the times this quantity is not known and measurements of its value are not available. In fact, in most applications, only the target position is measured.

In this work, the problem of estimating the position, velocity, and acceleration of a target maneuvering in the 3D space using only measurements of its position is addressed. A cascade of a parameter identifier and an $\mathcal{H}_2$ adaptive filter is used, where the first estimates the target angular speed and the second combines these estimates with measurements of the target position to estimate the target state.

The problem at hand could have been addressed resorting to other strategies, such as robust linear filtering, for instance. However, it is straightforward to show that any linear filter designed for the system considered in this work using a wrong model for the target dynamics, i.e., a wrong value for the target angular speed, will be biased. Other approaches inspired, for instance, in Lyapunov theory or backstepping, see [6], have also failed, since both strategies require the observation of the target velocity and acceleration, which are not available for measurement.

The main contributions of this work are:

1) a new $\mathcal{H}_2$ adaptive filter that estimates the position, linear velocity, and linear acceleration of a target using only position observations;
2) a new parameter identifier that estimates the (assumed constant) target angular speed — the structure of this identifier is slightly different from the usual approaches, since there is only one unknown parameter, but several measurements depending on its value;
3) a guarantee that, under persistence of excitation conditions, the error in the angular speed identification and the error in the state estimates provided by the adaptive filter converge exponentially fast to zero in the deterministic setup, i.e., in the absence of noise, and are bounded when bounded stochastic disturbances are considered and there is an upper bound on the target linear velocity and angular speed.

The proposed framework is not restricted to systems with sensors that provide direct measurements of the target position. It can be used with other sensors, such as RADAR or SONAR, see [1] and [7], as long as their measurements can be transformed into measurements of the target position.

This paper is organized as follows. The problem addressed in this work is formulated in section II, and the design and analysis of the online identification procedure that estimates the target angular speed are provided in section III. In section IV, an adaptive filter for the state of the target is derived using $\mathcal{H}_2$ strategies, and its stability and performance are discussed. Simulations illustrating the performance of the proposed estimators, in comparison with an Extended Kalman Filter (EKF), are presented in section V. Finally, in section VI, concluding remarks are provided.

NOMENCLATURE

In this paper, $|x|$ denotes the absolute value of the scalar x, $||\mathbf{x}||$ the Euclidean norm of the vector $\mathbf{x}$, and $||\mathbf{X}||$ the induced 2-norm of the matrix $\mathbf{X}$. If the vector $\mathbf{x}$ is a function

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of time in $\mathbb{R}^n$, $\mathbf{x} \in \mathcal{L}_2$ and $\mathbf{x} \in \mathcal{L}_\infty$ mean, respectively, that $||\mathbf{x}||_2 = (\int_0^\infty ||\mathbf{x}(t)||^2 \mathrm{d}t)^{1/2}$ and $||\mathbf{x}||_\infty = \sup_{t\geq 0}||\mathbf{x}(t)||$ are finite. The notation $\mathbf{X}_{i,j}$ is used to represent the entry of $\mathbf{X}$ in the i-th line and j-th column. The vector $\mathbf{e}_i$, $i = \{1, 2, 3\}$, denotes the i-th vector of the canonical basis of $\mathbb{R}^3$; tr$[\mathbf{X}]$ stands for the trace of a square matrix $\mathbf{X}$, and diag$[a_1, \ldots, a_n]$ corresponds to a diagonal matrix whose diagonal entries, starting in the upper left corner, are $a_1, \ldots, a_n$ (when these entries are matrices, the resulting matrix is block diagonal). The identity and zero matrices are denoted respectively by $\mathbf{I}_k$ and $\mathbf{0}_{m\times n}$, where k corresponds to the number of rows and columns of the identity matrix, and m and n correspond, respectively, to the number of rows and columns of the matrix of zeros. Finally, $\otimes$ denotes the Kronecker product and $\min[a, b]$ is used to denote the minimum of the two elements a and b.

II. Problem Formulation

The problem addressed in this paper is that of tracking and locating a target maneuvering in the three-dimensional space using observations of its position. The target position, linear velocity, and linear acceleration in the inertial (Cartesian) frame are denoted by $\mathbf{p} = [\mathrm{x}\ \ \mathrm{y}\ \ \mathrm{z}]^T$, $\mathbf{v} = [\dot{\mathrm{x}}\ \ \dot{\mathrm{y}}\ \ \dot{\mathrm{z}}]^T$, and $\mathbf{a} = [\ddot{\mathrm{x}}\ \ddot{\mathrm{y}}\ \ddot{\mathrm{z}}]^T$, respectively, where the dot represents the time derivative. Using this notation, the state $\mathbf{x} = [\mathrm{x}\ \dot{\mathrm{x}}\ \ddot{\mathrm{x}}\ \mathrm{y}\ \dot{\mathrm{y}}\ \ddot{\mathrm{y}}\ \mathrm{z}\ \dot{\mathrm{z}}\ \ddot{\mathrm{z}}]^T \in \mathbb{R}^9$ of the target is considered to evolve according to the *3D Planar Constant-Turn Model*

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\omega)\mathbf{x}(t) + \mathbf{B}\mathbf{d}(t), \tag{1}$$

as presented in [5], where

$$\mathbf{F}(\omega) = \mathrm{diag}\left[\,\overline{\mathbf{F}}(\omega), \overline{\mathbf{F}}(\omega), \overline{\mathbf{F}}(\omega)\,\right], \quad \overline{\mathbf{F}}(\omega) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\omega^2 & 0 \end{bmatrix},$$
$$\mathbf{B} = \mathbf{I}_3 \otimes \mathbf{b}, \qquad \mathbf{b} = \mathbf{e}_3,$$

and $\omega \geq 0$ is the (assumed constant, unknown, and bounded) target angular speed (norm of the target angular velocity vector). The process noise is denoted by $\mathbf{d}(t) \in \mathbb{R}^3$ and the time is represented by t. The three eigenvalues of $\overline{\mathbf{F}}(\omega)$ are 0, $-\omega j$, and ωj, where $j = \sqrt{-1}$ is the imaginary unit. Therefore, the nominal trajectories considered by this model are straight lines, parabolic trajectories, and ellipses.

The measurements $\mathbf{y}_m(t) \in \mathbb{R}^3$ of the position of the target with respect to the inertial reference frame are a linear function of the target state, and can be written in the form

$$\mathbf{y}_m(t) = \mathbf{p}(t) + \mathbf{D}\mathbf{n}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{n}(t), \tag{2}$$

where $\mathbf{n}(t) \in \mathbb{R}^3$ denotes the measurement noise, $\mathbf{C} = \mathbf{I}_3 \otimes \mathbf{e}_1^T$, and $\mathbf{D} = \mathbf{I}_3$. Both the process and observation noises are assumed to be stochastic disturbances with bounded values, i.e., $\beta_d = ||\mathbf{d}||_\infty$ and $\beta_n = ||\mathbf{n}||_\infty$ are finite.

A clear statement of the problem addressed in the remaining of this paper is presented next.

Problem statement 1: Consider a target maneuvering in the 3D space according to the model in (1), with constant, unknown, and bounded angular speed. Moreover, assume that measurements of the target position, as described in (2), are available. In this case, design two estimators, one for the target state and other for its angular speed, such that the errors in both cases i) converge exponentially fast to zero when no process and observation noises are present, and ii) are bounded when bounded noise is considered and there is an upper bound on the target linear velocity.

In order to solve this problem, a cascade of a parameter identifier and an adaptive filter, as depicted in Fig. 1, is proposed. In the figure, $\hat{\omega}$ and $\hat{\mathbf{x}}$ denote, respectively, the estimates of the target angular speed ω and the estimates of the target state $\mathbf{x}$.

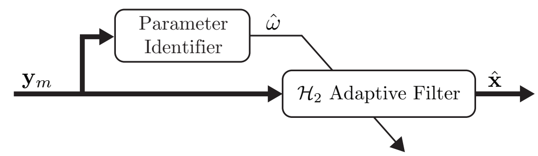


Fig. 1. Parameter identifier and adaptive filter interconnection.

III. Angular Speed Identification

In this section, the design and analysis in continuous-time of a parameter identifier that estimates the angular speed of a target moving according to the model in (1) are provided. This identifier resorts only to position measurements obtained as in (2), and builds on strategies commonly used in adaptive control, see [8] and [9].

From the model in (1), it is easy to conclude that $\dot{\mathbf{a}}(t) = \alpha\mathbf{v}(t) + \mathbf{d}(t)$, where $\alpha = -\omega^2$. By writing this relation as a function of the measurements $\mathbf{y}_m(t)$, the expression $\dddot{\mathbf{y}}_m(t) = \alpha\dot{\mathbf{y}}_m(t) + \dddot{\mathbf{n}}(t) - \alpha\dot{\mathbf{n}}(t) + \mathbf{d}(t)$ is obtained. Let s be the Laplace operator and $\mathbf{Y}_m(s) \in \mathbb{R}^3$, $\mathbf{D}(s) \in \mathbb{R}^3$, and $\mathbf{N}(s) \in \mathbb{R}^3$ be vectors where the i-th entry corresponds to the Laplace transform of the i-th entry of $\mathbf{y}_m(t)$, $\mathbf{d}(t)$, and $\mathbf{n}(t)$, respectively. Taking the unilateral Laplace transform of the previous expression, yields

$$s^3\mathbf{Y}_m(s) = \alpha s\mathbf{Y}_m(s) + (s^3 - \alpha s)\mathbf{N}(s) + \mathbf{D}(s) + \\ + s^2\mathbf{p}_0 + s\mathbf{v}_0 + \mathbf{a}_0 - \alpha\mathbf{p}_0,$$

where $\mathbf{p}_0$, $\mathbf{v}_0$, and $\mathbf{a}_0$ denote the initial values of $\mathbf{p}$, $\mathbf{v}$, and $\mathbf{a}$, respectively.

To avoid the use of differentiators, the entries of the vectors in the previous expression are filtered with a third-order stable filter $1/\Lambda(s)$, where $\Lambda(s)$ is a monic Hurwitz polynomial, e.g., $\Lambda(s) = (s+\lambda)^3$, $\lambda > 0$, which leads to

$$\underbrace{\frac{s^3}{\Lambda(s)}\mathbf{Y}_m(s)}_{\mathbf{\Psi}(s)} = \alpha\underbrace{\frac{s}{\Lambda(s)}\mathbf{Y}_m(s)}_{\mathbf{\Phi}(s)} + \underbrace{\frac{(s^3 - \alpha s)\mathbf{N}(s) + \mathbf{D}(s)}{\Lambda(s)}}_{\mathbf{\Xi}(s)} + \\ + \underbrace{\frac{s^2\mathbf{p}_0 + s\mathbf{v}_0 + \mathbf{a}_0 - \alpha\mathbf{p}_0}{\Lambda(s)}}_{\mathbf{Q}(s)}. \tag{3}$$

In this formula, $\mathbf{\Psi}(s)$ and $\mathbf{\Phi}(s)$ denote the Laplace transforms of the signals $\boldsymbol{\psi}(t) \in \mathbb{R}^3$ and $\boldsymbol{\phi}(t) \in \mathbb{R}^3$, whose entries are obtained by filtering each entry of $\mathbf{y}_m(t)$ with the causal linear time-invariant filters $s^3/\Lambda(s)$ and $s/\Lambda(s)$, with null initial conditions, i.e., by convolving each entry of $\mathbf{y}_m(t)$ with $h_{\boldsymbol{\psi}}(t)$ and $h_{\boldsymbol{\phi}}(t)$, where $h_{\boldsymbol{\psi}}(t)$ denotes the inverse Laplace transform of $s^3/\Lambda(s)$ and $h_{\boldsymbol{\phi}}(t)$ the inverse Laplace transform of $s/\Lambda(s)$. Moreover, $\mathbf{\Xi}(s)$ denotes the Laplace transform of the signal $\boldsymbol{\xi}(t) \in \mathbb{R}^3$, that results from filtering the process and observation noises with the filters $1/\Lambda(s)$ and $(s^3 - \alpha s)/\Lambda(s)$, respectively. The term $\mathbf{Q}(s)$ denotes the Laplace transform of the signal $\mathbf{q}(t) \in \mathbb{R}^3$, which comes from the initial conditions. The two terms $\boldsymbol{\xi}(t)$ and $\mathbf{q}(t)$ are not known since only measurements of the target position are available and they are a function of the unknown quantities $\mathbf{d}(t)$, $\mathbf{n}(t)$, $\mathbf{p}_0$, $\mathbf{v}_0$, $\mathbf{a}_0$, and α. However, it is straightforward to show that $||\mathbf{q}(t)||$ converges exponentially fast to zero, thus this term vanishes with time. Moreover, $||\boldsymbol{\xi}(t)||$ is bounded if the process and observation noises are bounded.

A. Angular speed adaptive law

In order to derive an adaptive law that provides estimates for α, consider the estimate $\hat{\boldsymbol{\psi}}(t)$ of $\boldsymbol{\psi}(t)$, with expression

$$\hat{\boldsymbol{\psi}}(t) = \hat{\alpha}(t)\boldsymbol{\phi}(t), \tag{4}$$

obtained resorting to an estimate $\hat{\alpha}(t)$ of the unknown parameter α, at time t. Since the value of α is unknown, the error $\tilde{\alpha}(t) = \alpha - \hat{\alpha}(t)$ in its estimation is not available. However, the estimation error $\boldsymbol{\varepsilon}(t) = (\boldsymbol{\psi}(t) - \hat{\boldsymbol{\psi}}(t))/m_\phi^2(t)$ can be generated resorting to the measurements available and reflects the difference between α and $\hat{\alpha}(t)$:

$$\boldsymbol{\varepsilon}(t) = \frac{\tilde{\alpha}(t)\boldsymbol{\phi}(t)}{m_\phi^2(t)} + \frac{\boldsymbol{\xi}(t)}{m_\phi^2(t)} + \frac{\mathbf{q}(t)}{m_\phi^2(t)}. \tag{5}$$

The new quantity $m_\phi^2(t)$ corresponds to a normalization signal that guarantees that the entries of $\boldsymbol{\phi}(t)/m_\phi(t)$ are bounded, and is sometimes used in the context of parameter identification, see examples in [8] and [9]. This property is useful in the analysis of the convergence of the estimates $\hat{\alpha}(t)$, to the real parameter α, when $\boldsymbol{\phi}(t)$ is not guaranteed to be bounded. In this work, the signal $m_\phi^2(t) = 1 + \mu\,\boldsymbol{\phi}^T(t)\boldsymbol{\phi}(t)$, $\mu > 0$, is considered.

Estimates $\hat{\alpha}(t)$ of the unknown parameter α can be obtained by minimizing the cost function

$$J(\hat{\alpha}(t)) = \frac{||\boldsymbol{\varepsilon}(t)||^2\, m_\phi^2(t)}{2} = \frac{||\boldsymbol{\psi}(t) - \hat{\alpha}(t)\boldsymbol{\phi}(t)||^2}{2\, m_\phi^2(t)}, \tag{6}$$

which depends quadratically on the estimation error $\boldsymbol{\varepsilon}(t)$. The minimization of this function with respect to $\hat{\alpha}(t)$ is performed resorting to the normalized (the normalization signal $m_\phi^2(t)$ is considered) gradient method $\dot{\hat{\alpha}}(t) = -\gamma\nabla J(\hat{\alpha}(t))$, where $\gamma > 0$ is a constant usually referred to as the adaptation gain and $\nabla J(\hat{\alpha}(t))$ is the gradient of $J(\hat{\alpha}(t))$ with respect to $\hat{\alpha}(t)$. The following adaptive law results

$$\dot{\hat{\alpha}}(t) = \gamma\boldsymbol{\varepsilon}^T(t)\boldsymbol{\phi}(t), \quad \hat{\alpha}(0) = \hat{\alpha}_0, \tag{7}$$

where $\hat{\alpha}_0$ denotes the initial estimate of α.

B. Angular speed convergence – deterministic framework

For convergence study purposes, let us start by considering a deterministic framework, i.e., consider that the process and observation noises introduced in section II are not present (the influence of these noises is addressed in section III-C). In this case, the proposed adaptive law ensures that the estimation error $\boldsymbol{\varepsilon}(t)$ converges to zero, but does not imply that $\hat{\alpha}(t)$ converges to α. In order to guarantee this property, some conditions must be imposed on $\boldsymbol{\phi}(t)$. These conditions are derived in Theorem 1, whose proof depends on Definition 1 and on Lemma 1.

Definition 1 ([10]): The linear state equation $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$, $\mathbf{x}(t_0) = \mathbf{x}_0$, is called uniformly exponentially stable (UES) if there exist finite positive constants γ_u, λ_u such that for any initial time instant t_0 and any initial condition $\mathbf{x}_0$, the corresponding solution satisfies $||\mathbf{x}(t)|| \leq \gamma_u e^{-\lambda_u(t-t_0)}||\mathbf{x}_0||$, $t \geq t_0$.

Lemma 1 ([11]): Consider the system $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{u}(t)$. If $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$ is UES and $||\mathbf{u}(t)||$ is exponentially decaying, then $||\mathbf{x}(t)||$ converges to zero exponentially fast.

In [8], stability and convergence guarantees for identification algorithms where one or more unknown parameters are considered, and $\boldsymbol{\psi}(t)$ is a scalar, can be found. In this work, these properties are generalized for cases where there is only one unknown parameter, but $\boldsymbol{\psi}(t)$ is a vector, see Theorem 1. The reasoning used to prove this generalization is completely different from the ones in [8].

Theorem 1: In the deterministic case, the identifier structure described previously, combined with the normalized gradient algorithm (7), guarantees that $\hat{\alpha}(t)$ converges to the nominal parameter α exponentially fast, if $||\overline{\boldsymbol{\phi}}(t)|| = ||\frac{\boldsymbol{\phi}(t)}{m_\phi(t)}||$ is persistently exciting.

Proof: Let the parameter estimation error be denoted by $\tilde{\alpha}(t) = \alpha - \hat{\alpha}(t)$. Since α is constant, when the process and observation noises are not considered we have

$$\dot{\tilde{\alpha}}(t) = -\gamma\boldsymbol{\varepsilon}^T(t)\boldsymbol{\phi}(t) = -\gamma\frac{||\boldsymbol{\phi}(t)||^2}{m_\phi^2(t)}\tilde{\alpha}(t) - \gamma\frac{\mathbf{q}^T(t)\boldsymbol{\phi}(t)}{m_\phi^2(t)}, \tag{8}$$

with $\tilde{\alpha}(t_0) = \tilde{\alpha}_0$, where t_0 denotes the initial time instant and $\tilde{\alpha}_0$ the initial parameter estimation error. Moreover, if $||\overline{\boldsymbol{\phi}}(t)||$ is persistently exciting (PE), then there exist $\theta_0 > 0$ and $T_0 > 0$ such that $\int_t^{t+T_0}||\overline{\boldsymbol{\phi}}(\tau)||^2\mathrm{d}\tau \geq \theta_0 T_0, \quad \forall t \geq 0$, see the definition of persistence of excitation in [8].

In order to prove this theorem, we are going to start by proving that the homogeneous part of the equation in (8) is UES, see Definition 1, if $||\overline{\boldsymbol{\phi}}(t)||$ is PE. With this purpose, consider the continuously differentiable function

$$V(t, \tilde{\alpha}(t)) = \int_t^{t+T_0}\tilde{\alpha}^2(\tau)\mathrm{d}\tau, \quad \forall\, t \geq 0. \tag{9}$$

Since the solution of the homogeneous equation is given by

$$\tilde{\alpha}(\tau) = \tilde{\alpha}(t)e^{-\gamma\int_t^\tau||\overline{\boldsymbol{\phi}}(\sigma)||^2\mathrm{d}\sigma}, \quad \tau \geq t, \tag{10}$$

the function in (9) can be written in the form

$$V(t, \tilde{\alpha}(t)) = \int_t^{t+T_0}\tilde{\alpha}^2(t)e^{-2\gamma\int_t^\tau||\overline{\boldsymbol{\phi}}(\sigma)||^2\mathrm{d}\sigma}\mathrm{d}\tau, \ \forall\, t \geq 0. \tag{11}$$

Moreover, $||\overline{\boldsymbol{\phi}}(t)||$ is bounded, i.e., $\beta = \sup_{\tau\geq 0}\left\|\overline{\boldsymbol{\phi}}(\tau)\right\|$ is a finite constant, thus $0 \leq \int_t^\tau||\overline{\boldsymbol{\phi}}(\sigma)||^2\mathrm{d}\sigma \leq \beta^2(\tau - t)$, $\tau \geq t$. Resorting to these inequalities and to the expression in (11), it is possible to conclude that

$$\frac{1 - e^{-2\gamma\beta^2T_0}}{2\gamma\beta^2}\tilde{\alpha}^2(t) \leq V(t, \tilde{\alpha}(t)) \leq T_0\tilde{\alpha}^2(t), \quad \forall\, t \geq 0.$$

From (9), the derivative of $V(t, \tilde{\alpha}(t))$ with respect to time is $\dot{V}(t, \tilde{\alpha}(t)) = \tilde{\alpha}^2(t + T_0) - \tilde{\alpha}^2(t)$. If $||\overline{\boldsymbol{\phi}}(t)||$ is assumed to be PE and (10) is used with $\tau = t + T_0$, it is straightforward to show that there exist $\theta_0 > 0$ and $T_0 > 0$ such that

$$\dot{V}(t, \tilde{\alpha}(t)) \leq -\left(1 - e^{-2\theta_0T_0}\right)\tilde{\alpha}^2(t), \quad \forall\, t \geq 0.$$

If such $\theta_0 > 0$ and $T_0 > 0$ are considered, then there exist positive constants $k_1 = (1 - e^{-2\gamma\beta^2T_0})/(2\gamma\beta^2)$, $k_2 = T_0$, and $k_3 = 1 - e^{-2\theta_0T_0}$, such that $k_1\tilde{\alpha}^2(t) \leq V(t, \tilde{\alpha}(t)) \leq k_2\tilde{\alpha}^2(t)$ and $\dot{V}(t, \tilde{\alpha}(t)) \leq -k_3\tilde{\alpha}^2(t)$, for all $t \geq 0$. Therefore, if $||\overline{\boldsymbol{\phi}}(t)||$ is PE, the homogeneous equation associated with the time-varying system in (8) is UES, see Theorem 4.10 in [12].

Since $||\mathbf{q}(t)||$ is exponentially decaying, $||\boldsymbol{\phi}(t)/m_\phi(t)||$ is bounded, and $m_\phi(t) \geq 1$, the norm of the term

$-\gamma\mathbf{q}^T(t)\boldsymbol{\phi}(t)/m_\phi^2(t)$, in (8), converges exponentially fast to zero. Therefore, according to Lemma 1, $|\tilde{\alpha}(t)|$ also converges to zero exponentially fast. ■

When $\mathbf{v}(0)$ and $\mathbf{a}(0)$ are not both null, the signal $||\overline{\boldsymbol{\phi}}(t)||$ is PE, which is easily understood by analyzing the trajectories that the model in (1) assumes for the target. Therefore, according to Theorem 1, in a deterministic framework $\hat{\alpha}(t)$ is guaranteed to converge to α exponentially fast unless $\mathbf{v}(0) = \mathbf{a}(0) = \mathbf{0}$, i.e., unless the target does not move, which was expected since trying to identify the target angular speed ω does not make sense in this situation.

C. Angular speed convergence – stochastic framework

When a stochastic framework is considered, i.e., when the process and observation noises, $\mathbf{d}(t)$ and $\mathbf{n}(t)$ respectively, introduced in section II are taken into account, the error $\tilde{\alpha}(t)$ associated with the estimation of the target angular speed cannot be expected to converge exactly to zero. However, it is possible to prove that this error converges to the vicinity of zero if some conditions are imposed on $\mathbf{d}(t)$, $\mathbf{n}(t)$, and $||\overline{\boldsymbol{\phi}}(t)||$. These conditions are stated in Theorem 2.

Theorem 2: If the process and observation noises, $\mathbf{d}(t)$ and $\mathbf{n}(t)$ respectively, are bounded and $||\overline{\boldsymbol{\phi}}(t)||$ is PE, then the normalized gradient algorithm (7) guarantees that there exist finite positive constants γ_1, λ_1, and $\beta_{\tilde{\alpha}}$ such that

$$|\tilde{\alpha}(t)| \leq \gamma_1 e^{-\lambda_1(t-t_0)} + \beta_{\tilde{\alpha}}, \quad \forall\, t \geq t_0. \tag{12}$$

Proof: This theorem is proven using linear systems theory and bounded-input, bounded-output stability principles, see [10]. Its proof is omitted here due to lack of space. ■

According to Theorem 2, when the process and observation noises are bounded, $||\overline{\boldsymbol{\phi}}(t)||$ is PE, and the initial transient is vanished, the norm of the error in the estimation of the unknown parameter verifies $|\tilde{\alpha}(t)| \leq \beta_{\tilde{\alpha}}$, which guarantees that the angular speed estimates converge to the vicinity of the target angular speed.

D. Gradient projection method

The parameter $\alpha = -\omega^2$ to be estimated cannot be positive. Therefore, instead of minimizing (6) for all $\hat{\alpha}(t) \in \mathbb{R}$, we want to constrain the estimation to be within the convex subset $\mathcal{S} \triangleq \{\hat{\alpha}(t) \in \mathbb{R} : \alpha_1 \leq \hat{\alpha}(t) \leq \alpha_2\}$ of $\mathbb{R}$, where $\alpha_1 \leq \alpha_2 \leq 0$. This is accomplished resorting to the gradient projection method, see [8] for details. According to this method, instead of the adaptive law in (7), the new law

$$\dot{\hat{\alpha}}(t) = \begin{cases} \gamma\boldsymbol{\varepsilon}^T(t)\boldsymbol{\phi}(t), & \text{if } \alpha_1 < \hat{\alpha}(t) < \alpha_2 \\ & \text{or if } \hat{\alpha}(t) = \alpha_1 \text{ and } \boldsymbol{\varepsilon}^T(t)\boldsymbol{\phi}(t) \geq 0, \\ & \text{or if } \hat{\alpha}(t) = \alpha_2 \text{ and } \boldsymbol{\varepsilon}^T(t)\boldsymbol{\phi}(t) \leq 0, \\ 0 & \text{, otherwise,} \end{cases} \tag{13}$$

is used. This adaptive law retains the properties derived in the absence of projection, while guaranteeing that $\hat{\alpha}(t) \in [\alpha_1,\, \alpha_2]$, for all t, as long as $\hat{\alpha}_0 \in \mathcal{S}$ and $\alpha \in \mathcal{S}$. The proof of this statement is omitted here due to space constraints.

Estimates $\hat{\omega}(t)$ for the target angular speed can be obtained from $\hat{\alpha}(t)$ as $\hat{\omega}(t) = \sqrt{-\hat{\alpha}(t)}$.

IV. $\mathcal{H}_2$ Adaptive Filter

In this section, a continuous-time $\mathcal{H}_2$ adaptive filter that estimates the state of a target moving according to the model in (1), resorting only to measurements of the target position and estimates of its angular speed, is proposed. The stability and performance of the filter are studied.

If, instead of the target angular speed ω, $\alpha = -\omega^2$ is considered, the model in (1) for the target can be written as an affine parameter dependent system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\alpha)\mathbf{x}(t) + \mathbf{B}\mathbf{d}(t),$$

where $\mathbf{A}(\alpha) = \operatorname{diag}\left[\overline{\mathbf{A}}(\alpha),\, \overline{\mathbf{A}}(\alpha),\, \overline{\mathbf{A}}(\alpha)\right] \in \mathbb{R}^{9\times 9}$ and

$$\overline{\mathbf{A}}(\alpha) = \underbrace{\left[\begin{array}{ccc} \mathbf{0}_{3\times 1} & \mathbf{e}_1 & \mathbf{e}_2 \end{array}\right]}_{\overline{\mathbf{A}}_0} + \alpha \underbrace{\left[\begin{array}{ccc} \mathbf{0}_{3\times 1} & \mathbf{e}_3 & \mathbf{0}_{3\times 1} \end{array}\right]}_{\overline{\mathbf{A}}_1}.$$

Moreover, consider that the target angular speed is bounded, i.e., that there exist $\alpha_1 \leq 0$ and $\alpha_2 \leq 0$ such that $\alpha \in [\alpha_1,\, \alpha_2]$. If estimates $\hat{\alpha}(t)$, obtained according to (13), of the value of α and measurements $\mathbf{y}_m(t)$, as defined in (2), of the target position are used, the following adaptive filter for the state $\mathbf{x}(t)$, with structure motivated by a linear filter, results

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}(\hat{\alpha}(t))\hat{\mathbf{x}}(t) + \mathbf{L}(\mathbf{y}_m(t) - \hat{\mathbf{y}}(t)),\ \hat{\mathbf{x}}(0) = \hat{\mathbf{x}}_0, \tag{14}$$

where $\hat{\mathbf{x}}(t)$ is an estimate of $\mathbf{x}(t)$, $\hat{\mathbf{y}}(t) = \mathbf{C}\hat{\mathbf{x}}(t)$, and $\hat{\mathbf{x}}_0$ denotes the filter initial conditions. The matrix $\mathbf{L} \in \mathbb{R}^{9\times 3}$ is the gain of the filter.

The dynamics of the state estimation error $\tilde{\mathbf{x}}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ associated with the filter can be written in the form

$$\begin{aligned} \dot{\tilde{\mathbf{x}}}(t) = {} & (\mathbf{A}(\alpha) - \mathbf{L}\mathbf{C} - \tilde{\alpha}(t)\mathbf{A}_1)\tilde{\mathbf{x}}(t) + \\ & + \tilde{\alpha}(t)\mathbf{A}_1\mathbf{x}(t) + \mathbf{B}\mathbf{d}(t) - \mathbf{L}\mathbf{D}\mathbf{n}(t), \end{aligned} \tag{15}$$

where $\mathbf{A}_1 = \operatorname{diag}\left[\overline{\mathbf{A}}_1,\, \overline{\mathbf{A}}_1,\, \overline{\mathbf{A}}_1\right] \in \mathbb{R}^{9\times 9}$.

A. Filter stability

In a deterministic framework, i.e., when the process and observation noises introduced in section II are not present, conditions on the gain $\mathbf{L}$ that ensure that the error of the filter in (14) converges exponentially fast to zero can be imposed. These conditions are provided in Theorem 3.

Theorem 3: When a deterministic framework is considered and $||\overline{\boldsymbol{\phi}}(t)||$ is persistently exciting, the error of the filter in (14), with $\hat{\alpha}(t)$ computed resorting to (13) and gain $\mathbf{L}$ chosen to guarantee that both $\dot{\tilde{\mathbf{x}}}(t) = [\mathbf{A}(\alpha_1) - \mathbf{L}\mathbf{C}]\,\tilde{\mathbf{x}}(t)$ and $\dot{\tilde{\mathbf{x}}}(t) = [\mathbf{A}(\alpha_2) - \mathbf{L}\mathbf{C}]\,\tilde{\mathbf{x}}(t)$ are UES for given values of $\alpha_1 \leq 0$ and $\alpha_2 \leq 0$, converges to zero exponentially fast.

When both the process and observation noises are considered, it is possible to prove that the filter estimation error converges to the vicinity of zero and that, after the initial transient, its maximum norm has an upper bound if some conditions are imposed on $\mathbf{L}$, $\mathbf{d}(t)$, $\mathbf{n}(t)$, $||\overline{\boldsymbol{\phi}}(t)||$, and on the target maximum linear velocity. These conditions are stated in Theorem 4.

Theorem 4: Consider the filter in (14), with $\hat{\alpha}(t)$ computed resorting to (13) and gain $\mathbf{L}$ chosen to guarantee that both $\dot{\tilde{\mathbf{x}}}(t) = [\mathbf{A}(\alpha_1) - \mathbf{L}\mathbf{C}]\,\tilde{\mathbf{x}}(t)$ and $\dot{\tilde{\mathbf{x}}}(t) = [\mathbf{A}(\alpha_2) - \mathbf{L}\mathbf{C}]\,\tilde{\mathbf{x}}(t)$ are UES for given values of $\alpha_1 \leq 0$ and $\alpha_2 \leq 0$. Moreover, assume that $||\overline{\boldsymbol{\phi}}(t)||$ is persistently exciting and that the process noise $\mathbf{d}(t)$, the observation noise $\mathbf{n}(t)$, and the target linear velocity $\mathbf{v}(t)$ are bounded. In this case, there exists a finite positive constant $\beta_{\tilde{\mathbf{x}}}$ such that, after an initial transient, the filter estimation error verifies

$$||\tilde{\mathbf{x}}(t)|| \leq \beta_{\tilde{\mathbf{x}}}, \quad \forall\, t \geq t_0. \tag{16}$$

It is possible to show that a gain $\mathbf{L}$ verifying the constraints imposed on this quantity in Theorems 3 and 4 can always be found. The proof of this statement and the proofs of these two theorems are omitted here due to lack of space.

B. Design of the gain of the $\mathcal{H}_2$ filter

The gain of the filter is obtained using $\mathcal{H}_2$ and LMI-based methods, see [13] and [14] for details about the use of LMIs in the design of $\mathcal{H}_2$ filters. In order to design this gain, rewrite the dynamics in (14) in the form

$$\dot{\hat{\mathbf{x}}}(t)=\mathbf{A}(\alpha)\hat{\mathbf{x}}(t)+\mathbf{L}(\mathbf{y}_m(t)-\hat{\mathbf{y}}(t))-\mathbf{A}_1\tilde{\alpha}(t)\hat{\mathbf{x}}(t),\ \hat{\mathbf{x}}(0)=\hat{\mathbf{x}}_0.$$

This estimator has the structure of a linear filter, but with the extra term $\mathbf{A}_1\tilde{\alpha}(t)\hat{\mathbf{x}}(t)$, which will be treated as a disturbance in this section, since it depends on the unknown time-varying quantity $\tilde{\alpha}(t)$, whose impact on the filter performance we want to minimize. This term can be rewritten in the form $\tilde{\alpha}(t)\mathbf{B}\hat{\mathbf{v}}(t)$, where $\hat{\mathbf{v}}(t)$ corresponds to the target linear velocity estimates, i.e., $\hat{\mathbf{v}}(t)=[\hat{\mathbf{x}}_2(t)\ \hat{\mathbf{x}}_5(t)\ \hat{\mathbf{x}}_8(t)]^T$, where $\hat{\mathbf{x}}_k(t)$ denotes the k-th entry of $\hat{\mathbf{x}}(t)$.

When the disturbance introduced in the previous paragraph is considered, together with the process noise $\mathbf{d}(t)$ and the measurement noise $\mathbf{n}(t)$, the dynamics of the error in (15) can be written in the form

$$\dot{\tilde{\mathbf{x}}}(t)=(\mathbf{A}(\alpha)-\mathbf{LC})\,\tilde{\mathbf{x}}(t)+\mathbf{B}[\mathbf{d}(t)+\tilde{\alpha}(t)\hat{\mathbf{v}}(t)]-\mathbf{LDn}(t).$$

The term $\tilde{\alpha}(t)\hat{\mathbf{v}}(t)\in\mathbb{R}^3$ and the process noise $\mathbf{d}(t)$ affect the estimation error $\tilde{\mathbf{x}}(t)$ in the same way (through $\mathbf{B}$), i.e., they both corrupt directly the error associated with the target acceleration estimates. Therefore, for design purposes, a single disturbance term $\boldsymbol{\delta}(t)=\mathbf{d}(t)+\tilde{\alpha}(t)\hat{\mathbf{v}}(t)$ is considered to model the effects of these two quantities. By concatenating this disturbance with the noise that corrupts the measurements of the target position into a single vector, the following generalized disturbance vector results $\mathbf{w}(t)=[\boldsymbol{\delta}^T(t)\ \ \mathbf{n}^T(t)]^T\in\mathbb{R}^6$. Rewriting the dynamics of the error as a function of this generalized disturbance yields

$$\dot{\tilde{\mathbf{x}}}(t)=(\mathbf{A}(\alpha)-\mathbf{LC})\,\tilde{\mathbf{x}}(t)+(\mathbf{B}_w-\mathbf{LD}_{yw})\,\mathbf{w}(t), \quad (17)$$

where $\mathbf{D}_{yw}=[\mathbf{0}_{3\times 3}\ \ \mathbf{D}]$ and $\mathbf{B}_w=[\mathbf{B}\ \ \mathbf{0}_{9\times 3}]$.

If only the target position estimation error $\mathbf{e}(t)=\mathbf{C}\tilde{\mathbf{x}}(t)\in\mathbb{R}^3$ is considered for performance purposes, the gain $\mathbf{L}$ that minimizes the $\mathcal{H}_2$ norm $\|F_L\|_2$ of the system obtained from $\mathbf{w}(t)$ to $\mathbf{e}(t)$ can be computed using the strategies described in [14]. Since α is unknown, these strategies are used with two modified versions of (17), obtained by replacing α by α_1 and α_2. This is a standard approach that provides an upper bound for $\|F_L\|_2$ if $\alpha\in[\alpha_1,\alpha_2]$, see [15]. It is easy to show that the gain computed using this method verifies the stability constraints imposed in Theorems 3 and 4.

V. Simulation Results

In this section, continuous-time simulation results illustrating the performance of the proposed parameter identification procedure and adaptive filter are presented.

For comparison purposes, results obtained with an Extended Kalman Filter, see [16] for details, are also provided. This filter was designed for the nonlinear system that results from augmenting the state $\mathbf{x}\in\mathbb{R}^9$, of (1), with the target angular speed ω. The new state variable was modeled as a Wiener process [5]. The model considered for the measurements was the one introduced in (2).

In this section, measurements obtained with a single PTZ (pan, tilt, and zoom) camera are considered. In particular, measurements of the center of the target in the images and measurements of its distance to the camera, which can be obtained using the strategies proposed in [17], for instance, are used. These (range and bearing) observations can be transformed to Cartesian coordinates using a nonlinear transformation, see examples in [1], which leads to the model in (2) for the position measurements.

In the simulations presented in this section, the intrinsic parameters of a 215 PTZ camera from AXIS are used. The inertial reference frame is considered to have its origin in the camera optical center, with the z-axis aligned with the optical axis of the camera in the beginning of the experiments. This is without loss of generality, since the placement of the inertial reference frame in some other location can be tackled by determining the camera extrinsic parameters during calibration, see [18] for details about camera models and calibration procedures. The target angular speed is considered to belong to the interval $[0,\,0.5]$ rad/s, thus $\alpha_1=-0.25$ and $\alpha_2=0$. In the online angular speed identification procedure, the parameters $\mu=\gamma=10^{-10}$ and the Hurwitz polynomial $\Lambda(s)=(s+\lambda)^3$, $\lambda=0.2$, are used. The gain $\mathbf{L}$ of the $\mathcal{H}_2$ adaptive filter was computed using the strategy described in section IV-B and ensures that $\|F_L\|_2< 199.445$. Its non-null entries are $\mathbf{L}_{1,1}=\mathbf{L}_{4,2}=\mathbf{L}_{7,3}=1.33$, $\mathbf{L}_{2,1}=\mathbf{L}_{5,2}=\mathbf{L}_{8,3}=0.77$, and $\mathbf{L}_{3,1}=\mathbf{L}_{6,2}=\mathbf{L}_{9,3}=0.13$. In the design of this gain, the vector $\mathbf{b}=\mathbf{e}_3$ and the matrix $\mathbf{D}=\mathbf{I}_3$, introduced in section II, were replaced by $\mathbf{b}=b_d\,\mathbf{e}_3$ and $\mathbf{D}=d_n\mathbf{I}_3$, where b_d and d_n are positive constants. This strategy allows us to consider different intensities for the process and observation noises, $\mathbf{d}(t)$ and $\mathbf{n}(t)$, by choosing the values of b_d and d_n. In this case, the parameters $b_d=10$ and $d_n=100$ were used.

The measurements of the center of the target in the images and the measurements of its distance to the camera are corrupted by uniformly distributed noise, with values in the intervals $[-10,\,10]$ pixel and $[-1,\,1]$ m, respectively. In the design of the EKF, the process noise that affects the target acceleration and the measurement noise that corrupts the target position measurements are considered to have power spectral density matrices $10^2\mathbf{I}_3$ mm^{2}Hz^5 and $100^2\mathbf{I}_3$ mm^{2}Hz^{-1}, respectively. The power spectral density considered for the noise that affects the target angular speed is 10^{-6} rad^{2}Hz^3.

In the sequel, two experiments are reported. The first illustrates the performance of the proposed estimators when the target moves along a straight line with $\omega=0$ rad/s, and the second illustrates their performance when the target angular speed varies over time. The trajectories described by the target in the two situations are shown in Fig. 2.

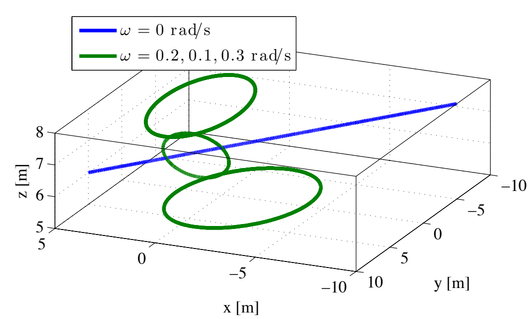
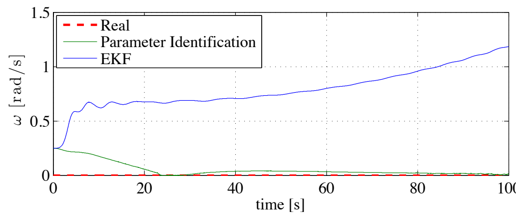
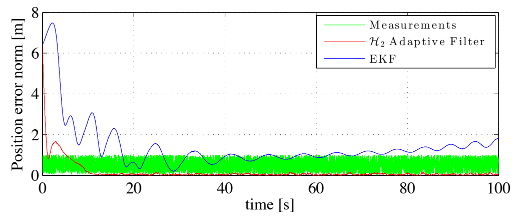


Fig. 2. Trajectories described by the target.

In Fig. 3(a), the target angular speed estimates provided by the identification procedure proposed in section III and by the EKF, for the first experiment, are depicted. As can be seen, the estimates provided by the parameter identifier converge to the vicinity of the target real angular speed $\omega = 0$ rad/s, whereas the EKF diverges.



(a) Angular speed identification.

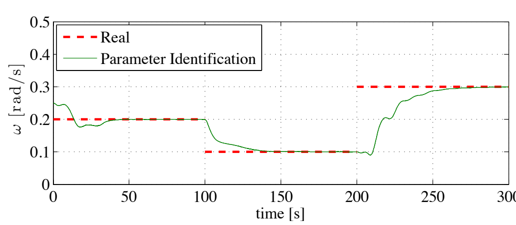


(b) Position estimation.

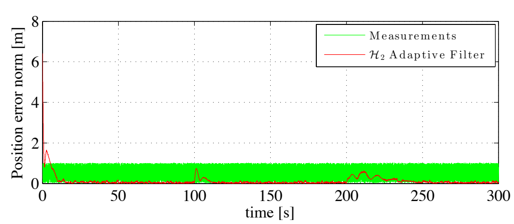
Fig. 3. Performance analysis for $\omega = 0$ rad/s.

The results obtained with the $\mathcal{H}_2$ adaptive filter in the first experiment are depicted in Fig. 3(b). These results are compared with the estimates provided by the EKF and with the measurements of the target position computed resorting to the aforementioned nonlinear transformation. As expected from the performance of the EKF in the estimation of the target angular speed, its estimates for the target position diverge. Even though the EKF diverges, the error in the estimates provided by the adaptive filter, and the error in the estimation of the target angular speed, converge to the vicinity of zero. These results are in accordance with Theorems 2 and 4. Moreover, the steady-state performance of the adaptive filter is significantly better than the one obtained with the measurements of the target position.

The results for the second experiment, in which the target moves along a trajectory with three different angular speeds, are presented in Fig. 4. As can be seen, the angular speed



(a) Angular speed identification.



(b) Position estimation.

Fig. 4. Performance analysis for a target with changing angular speed.

identification strategy proposed in section III is robust to variations in the parameter to be estimated, since the angular speed estimates converge to the target real angular speed even after abrupt changes in its value. The degradation in the performance of the position estimates obtained with the $\mathcal{H}_2$ filter, around time instants 100 s and 200 s, is due to the transients in the estimates provided by the parameter identifier when the target changes its angular speed.

VI. Conclusions

In this work, the problem of estimating the position, linear velocity, and linear acceleration of a target maneuvering in the 3D space was addressed. A model for the target that depends on its angular speed was considered and only measurements of the target position were used. This problem was tackled resorting to a cascade of a parameter identifier, which estimates the angular speed of the target, and an $\mathcal{H}_2$ adaptive filter, which combines the angular speed estimates with measurements of the target position to estimate the target state. Under persistence of excitation conditions and for experiments where the process noise, the observation noise, the target linear velocity, and the target angular speed are bounded, the errors associated with the proposed estimators were proved to converge to the vicinity of zero. Simulations showing that the convergence and stability guarantees derived in the paper hold, even when the estimates provided by an Extended Kalman Filter diverge, were presented.

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