Multiple Model Adaptive Estimation
for Open Loop Unstable Plants*

Vahid Hassani[1], António M. Pascoal[2], A. Pedro Aguiar[3] and Michael Athans[2&4]

Abstract— This paper addresses the problem of adaptive state and parameter estimation of open loop unstable plants using a multiple model structure. A state estimate is obtained as a probabilistically weighted sum of the estimates produced by a bank of individual observers. Model identification and convergence of the dynamic weights in the Multiple Model Adaptive Estimation (MMAE) for open-loop unstable plants are analyzed and the effect of the control action (by a controller in the loop) is studied. In the present paper we show that the techniques introduced in MMAE for open-loop stable plants and in the absence of control action are applicable to open-loop unstable plants with a stabilizing controller in the loop. A distance-like pseudo norm between the true plant and the identified model is developed and furthermore we show that the model identified is the one that is the closest to the true plant model in the defined norm among all models in the bank. The performance and convergence of the MMAE procedure are illustrated with Monte-Carlo simulation runs using the model of an inverted pendulum installed on a system of masses, springs, and dampers.

I. INTRODUCTION

The design of a single state-observer for a given plant requires exact knowledge of the plant parameters for superior performance. In practice, parameter uncertainty will impact the performance and robustness of the observer. In fact, incorrect modeling in the observer design may lead to large estimation errors or even error divergence [1]. To cope with this problem, adaptive estimation algorithms (where the adaptation is with respect to the uncertainty in the plant parameters) have been proposed in the literature. Among these, Multiple Model Adaptive Estimation (MMAE) algorithms have received special attention [2]–[4]. Notice however that the use of multiple models for Adaptive Estimation goes back to the 1960s and 1970s when several authors including [2], [3], studied Kalman filter based estimators.

In the stochastic version of the MMAE [2]–[4], a separate discrete-time Kalman filter (KF) is developed for each "selected model" (SM) defined by an hypothesized parameter vector in the unknown parameter set. The resulting set of

[1]Vahid Hassani was formerly with LARSyS, Portugal, and is now with the Norwegian Marine Technology Research Institute (MARINTEK), Trondheim, Norway. vahid.hassani@marintek.sintef.no
[2]The Laboratory of Robotics and Systems in Engineering and Science (LARSyS), Instituto Superior Técnico (IST), Univ. Técnica de Lisboa (UTL), Lisbon, Portugal. {antonio,michael}@isr.ist.utl.pt
[3]A. Pedro Aguiar is with the LARSyS and the Dept. of Electrical and Computer Engineering, Faculty of Engineering, University of Porto (FEUP), Porto, Portugal. pedro.aguiar@fe.up.pt
[4]Michael Athans is also Professor of EECS (emeritus), M.I.T., USA.

KFs forms a "bank" where each local KF generates its own state estimate and an output error (residual), as shown in Fig. 1. The bank of KFs runs in parallel and at each sampling instant the MMAE uses a nonlinear function of the measurement residuals of each SM to compute the conditional (a posterior) probability p_i that the filter selected be the one corresponding to the true plant model. The state estimate is a probabilistically weighted combination of each KF estimate. The rationale is that the highest probability should be assigned to the state estimation provided by the most accurate KF, and lower probabilities assigned to the remaining KFs.

In the last decade, MMAE have been the subject of considerable research effort that is patent in a vast number of publications; see [5]–[7] and the references therein. MMAE is at the root of many techniques for estimation, navigation, tracking, and surveillance. It is also the basis for Multiple-Model Adaptive Control, see [6], [8]–[10]. However, as far as we could ascertain the work reported is limited to open loop stable plants and fails therefore to address the effect of a stabilizing controller in the loop.

In [4], by introducing an information theoretic measure, the authors analyzed the convergence of the conditional probabilities p_i and showed that the one corresponding to the KF designed for the closest to the actual system (in a stochastic norm sense) converges to one, while the others tend to zero. The theoretical setup exploited in [4] is limited to open loop stable plants in the absence of control action. Similar results for open-loop stable plants in the absence of control action are derived in [3] using the Kullback information measure.

Fig. 1. The MMAE architecture

The main contribution of this paper is the extension of the MMAE structure to open loop unstable plants, showing that the a posterior probability corresponding to the model closest to the true plant converges to one. Moreover, we develop a distance-like pseudo norm between the true plant and the identified model (which depends on the controller that is used in the loop).

The structure of the paper is as follows. In section II we review the main issues of MMAE and define the structure of a standard MMAE. Section III summarizes our main results. Section IV illustrates the performance of the MMAE algorithm proposed through computer simulations with a model of an inverted pendulum installed on a Mass-Spring-Dashpot mechanical system. The conclusions are summarized in section V.

II. THE MULTIPLE-MODEL ADAPTIVE ESTIMATOR

This section introduces a class of MMAEs in a stochastic setting. A MMAE relies on a finite number N of selected models chosen from the original set of (possibly infinite) plant models and consists of: i) a Posterior Probability Evaluator (PPE) of N weighting signals and ii) a bank of N discrete-time observers, where each observer is designed based on one of the selected models adopted. The conditional probabilities are provided by a discrete-time dynamic equation called Posterior Probability Evaluator (PPE). Fig. 1 shows the structure of the MMAE in which the plant is described by a LTI discrete-time equation, $w(t)$ and $v(t)$ are plant disturbance and measurement noise sequences respectively, and the observers are designed using different values of the uncertain parameters. We assume that the plant model is subjected to parameter uncertainty $\theta \in \mathbb{R}^l$. In what follows we consider multiple-input-multiple-output (MIMO) linear time-invariant (LTI) discrete plant models of the form

$$x(t+1) = A_\theta x(t) + B_\theta u(t) + G_\theta w(t), \qquad (1)$$
$$y(t) = C_\theta x(t) + v(t),$$

where $x(t) \in \mathbb{R}^n$ denotes the state of the system, $u(t) \in \mathbb{R}^m$ its control input, $y(t) \in \mathbb{R}^q$ its measured noisy output, $w(t) \in \mathbb{R}^r$ an input plant disturbance that can not be measured, and $v(t) \in \mathbb{R}^q$ is the measurement noise. The vectors $w(t)$ and $v(t)$ are zero-mean, mutually independent white Gaussian sequences, with covariances $cov[w(t); w(\tau)] = Q\delta_{t\tau}$ and $cov[v(t); v(\tau)] = R\delta_{t\tau}$, respectively. The initial condition $x(0)$ of (1) is a Gaussian random vector with mean and covariance given by $E\{x(0)\} = 0$ and $E\{x(0)x^T(0)\} = P(0)$. The matrices A_θ, B_θ, G_θ, and C_θ contain *unknown constant parameters* denoted by the vector $\theta \in \Theta \subset \mathbb{R}^l$ where Θ is some compact set.

Consider a finite set of candidate parameter values $\boldsymbol{\vartheta} := \{\theta_1, \theta_2, \ldots, \theta_N\}$ indexed by $i \in \{1, \ldots, N\}$. We propose the following MMAE. The state, output, and parameter estimates are given in the form of

$$\hat{z}(t) := \sum_{i=1}^{N} p_i(t)\hat{z}_i(t), \qquad (2)$$

where the pair ($\hat{z}(t)$ and $\hat{z}_i(t)$) shall be replaced by ($\hat{x}(t)$ and $\hat{x}_i(t)$), ($\hat{y}(t)$ and $\hat{y}_i(t)$), or ($\hat{\theta}(t)$ and $\hat{\theta}_i(t)$). The estimates of the state $x(t)$, output $y(t)$, and parameter vector θ at time t, are denoted by $\hat{x}(t)$, $\hat{y}(t)$ and $\hat{\theta}(t)$, respectively. The variables $p_i(t)$ are conditional probabilities (which are defined below). In (2), each $\hat{x}_i(t)$; $i = 1, \ldots, N$ corresponds to a "local" state estimate generated by the ith (steady state) Kalman filter [3]

$$\hat{x}_i(t+1) = A_{\theta_i}\hat{x}_i(t) + B_{\theta_i}u(t) + H_{\theta_i}\big(y(t) - C_{\theta_i}\hat{x}_i(t)\big), \qquad (3a)$$
$$\hat{y}_i(t) = C_{\theta_i}\hat{x}_i(t), \qquad (3b)$$
$$H_{\theta_i} = A_{\theta_i}P_iC_{\theta_i}^T[C_{\theta_i}P_iC_{\theta_i}^T + R]^{-1} \qquad (3c)$$

where P_i is the solution of the discrete Riccati equation

$$\begin{aligned} P_i &= A_{\theta_i}P_iA_{\theta_i}^T + L_{\theta_i}QL_{\theta_i}^T \\ &\quad - A_{\theta_i}^T P_i C_{\theta_i}^T[C_{\theta_i}P_iC_{\theta_i}^T + R]^{-1}C_{\theta_i}P_iA_{\theta_i}, \end{aligned} \qquad (4)$$

where it is assumed that $[A_{\theta_i}, G_{\theta_i}]$ and $[A_{\theta_i}, C_{\theta_i}]$; $i = 1, \ldots, N$ are controllable and observable, respectively. The symmetric positive definite matrices Q and R are the covariance matrices of the plant disturbance and measurement noise, respectively. In the sequel we introduce the dynamics for the weights in (2).

A. Posterior Probability Evaluator (PPE)

The key to the MMAE algorithm is the so-called *posterior probability evaluator (PPE)* which evaluates, in real time, the a posterior (conditional) probability that each model generates the data, i.e. the probability that $\theta = \theta_i$; $i \in \{1, \ldots, N\}$. Thus, the PPE together with the bank of KFs represent the identification subsystem.

The a posterior probabilities can be computed on-line by the PPE using the recursive formula

$$p_i(t+1) = \frac{\beta_i e^{-w_i(t+1)}}{\sum_{j=1}^{N} p_j(t)\beta_j e^{-w_j(t+1)}} p_i(t), \qquad (5)$$

where $p_i(0)$ are the prior model probabilities and $w_i(t)$ and β_i are defined as

$$w_i(t) := \frac{1}{2}[y(t) - \hat{y}_{\theta_i}(t)]^T S_i^{-1}[y(t) - \hat{y}_{\theta_i}(t)], \qquad (6a)$$
$$\beta_i := \frac{1}{(2\pi)^{\frac{q}{2}}\sqrt{|S_i|}}, \qquad (6b)$$

where q is the dimension of the measurement vector $y(t)$ and $S_i = C_{\theta_i}P_iC_{\theta_i}^T + R$ is the covariance matrix of the residuals of the ith KF.

Equation (5), which generates the time-sequence of the a posterior probabilities p_i, arises from the application of Bayes rule (see [3], [11]). We impose the constraint that the initial conditions $p_i(0)$ be chosen such that $p_i(0) \in (0, 1)$ and $\sum_{i=1}^{N} p_i(0) = 1$ for obvious reasons.

III. MAIN RESULTS

To the best of our knowledge, previous work on MMAE is restricted to open-loop stable plant models. Furthermore, the effect of the control action has not been studied. In [4], Baram and Sandal proved that the output estimation errors

of the individual KFs in the MMAE structure, for open-loop stable plants in the absence of control action, are ergodic and stationary. Then, under the ergodicity and stationarity condition, they showed that the conditional probability p_i corresponding to the KF designed for the closest model to the actual system (in a well defined norm sense) converges to one, while the others tend to zero.

In this section, we study the convergence of the a posterior probabilities assigned to the individual KFs in the MMAE structure for unstable plants with a stabilizing controller in the feedback loop. For the time being, let us assume that the output estimation error residuals, $\tilde{y}_i(t) = y(t) - \hat{y}_i(t)$, are stationary and ergodic.[1] We will soon verify these assumptions.

Assumption (1): We will assume that the innovation (residual) sequences in all the KFs are stationary and ergodic.

Let $Y_t \equiv \{y(0), y(1), \cdots, y(t)\}$ condense the history of the measurements from the beginning up to time t. Consider the conditional probability density function $f_i\big(y(t)|Y_{t-1}, \theta_i\big)$. For each KF we have $f_i\big(Y_t|\theta_i\big) = \prod_{k=1}^{t} f_i\big(y(k)|Y_{k-1}, \theta_i\big)$. We will denote $p(\theta_i|Y_k)$ (shorthand for $p(\theta = \theta_i|Y_k)$) as the *a posteriori probabilities*. It is for the recursive calculation of these quantities that the bank of conditional KFs comes into play. The following equation, which is at the root of (5) applies:

$$p(\theta_i|Y_k) = \frac{f_i\big(y(t)|Y_{t-1}, \theta_i\big)}{\displaystyle\sum_{j=0}^{N} f_j\big(y(t)|Y_{t-1}, \theta_j\big) p(\theta_j|Y_{k-1})} p(\theta_i|Y_{k-1})$$

(7)

For two different KFs based on θ_i and θ_j, if

$$f_j\big(Y_t|\theta_j\big) > f_i\big(Y_t|\theta_i\big),$$

(8)

or, equivalently, if

$$\log f_j\big(Y_t|\theta_j\big) > \log f_i\big(Y_t|\theta_i\big),$$

we will say that the jth KF is preferred over (more likely or probable than) the ith KF based on the observation vector Y_t. Define the likelihood ratio for the sequence of Y_t

$$k_i^j\big(Y_t\big) = \frac{f_j\big(Y_t|\theta_j\big)}{f_i\big(Y_t|\theta_i\big)}$$

(9)

or, equivalently,

$$\log k_i^j\big(Y_t\big) = \log f_j\big(Y_t|\theta_j\big) - \log f_i\big(Y_t|\theta_i\big),$$

where $\log k_i^j\big(Y_t\big)$ can be regarded as a measure of the information contained in Y_t that can be used to select between j^{th} and i^{th} KFs.[2] Similarly, one can compute the conditional likelihood ratio

$$k_i^j\big(y(t)|Y_{t-1}\big) = \frac{f_j\big(y(t)|Y_{t-1}, \theta_j\big)}{f_i\big(y(t)|Y_{t-1}, \theta_i\big)}$$

(10)

or, equivalently,

$$\log k_i^j\big(y(t)|Y_{t-1}\big)$$
$$= \log f_j\big(y(t)|Y_{t-1}, \theta_j\big) - \log f_i\big(y(t)|Y_{t-1}, \theta_i\big)$$

which can be interpreted as a measure of the information contained in $y(t)$ that can be used to select between j^{th} and i^{th} KFs. We can define the mean information in $y(t)$ for preferring the jth KF over the ith KF as

$$d_t(j, i) = E\{\log k_i^j\big(y(t)|Y_{t-1}\big)\}.$$

(11)

When $d_t(j, i)$[3] is positive we can conclude that the jth KF is more probable to be the true KF than the ith KF. The above variable can be regarded as a yardstick against which to select the "best" KF trough the bank. It is easy to see that the true KF is always preferred over other KFs.

The conditional probability density of $y(t)$ given the past observation Y_{t-1} when θ_i is the true parameter is

$$f_i\big(y(t)|Y_{t-1}, \theta_i\big) = \frac{exp\{-\frac{1}{2}\tilde{y}_i(t)^T S_i^{-1} \tilde{y}_i(t)\}}{\sqrt{(2\pi)^q |S_i|}},$$

(12)

where q is the dimension of $\tilde{y}(t)$ and $S_i = C_{\theta_i} P_{\theta_i} C_{\theta_i}^T + R$ is the covariance of the innovation sequence. In fact, in this case the conditional probability density of $y(t)$ given the past observation Y_{t-1} when θ_i is the true parameter, $f_i\big(y(t)|Y_{t-1}, \theta_i\big)$, is a gaussian distribution with mean $\hat{y}_i(t)$ and covariance $E\{\tilde{y}_i(t)\tilde{y}_i^T(t)\}$, which we denote by S_i.[4]

Let us denote by $\theta_\star$ the true parameter in the plant; for each KF in the bank (not necessarily the true one) we have:

$$E \log\{f_i\big(y(t)|Y_{t-1}, \theta_i\big)\}$$

(13)

$$= -\frac{q}{2}\log(2\pi) - \frac{1}{2}\log(|S_i|) - \frac{1}{2}tr(S_i^{-1} E\{\tilde{y}_i^T(t)\tilde{y}_i(t)\})$$

$$= -\frac{q}{2}\log(2\pi) - \frac{1}{2}\log(|S_i|) - \frac{1}{2}tr(S_i^{-1} S_i^\star)$$

where S_i^j is the covariance of output estimation sequence when the true parameter in the plant is θ_j but the KF is designed based on θ_i.[5]

Now, it is easy to write $d(j, i)$ as

$$d(j, i) = $$

(14)

$$+ E \log\{f_j\big(y(t)|Y_{t-1}, \theta_j\big)\} - E \log\{f_i\big(y(t)|Y_{t-1}, \theta_i\big)\}$$

$$+ \frac{1}{2}\log(|S_i|) + \frac{1}{2}tr(S_i^{-1} S_i^\star) - \frac{1}{2}\log(|S_j|) - \frac{1}{2}tr(S_j^{-1} S_j^\star).$$

Let

$$\Gamma_i^\star \equiv \frac{1}{2}\log(|S_i|) + \frac{1}{2}tr(S_i^{-1} S_i^\star),$$

(15)

from which it follows that

$$d(j, i) = \Gamma_i^\star - \Gamma_j^\star.$$

(16)

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[1] We assume that a controller that stabilizes the open-loop unstable plant for all values of the parametric uncertainty set is in the feedback loop.

[2] Positive values of $\log k_i^j\big(Y_t\big)$ mean that the jth KF is more likely to be the optimal observer than the ith KF, based on the observation vector Y_t, while negative values show that the ith KF is preferred over the jth KF.

[3] When the uncertain parameter θ is constant, it is reasonable to assume that in steady state $y(t)$ and $y(\tau)$, $t \neq \tau$ have the same "amount" of information for selecting between the KFs. So we drop the t in $d_t(j, i)$ and use $d(j, i)$ instead.

[4] According to the assumption of stationarity, S_i is independent of t.

[5] We should highlight here that the notation of the term $\tilde{y}_i(t)$ in (13) is ambiguous, since it may denote either the residual of the ith KF designed based on the assumption that for the true plant $\theta = \theta_i$, or the residual of the ith KF irrespective of the true value of θ in the plant. Clearly, in (13) $\tilde{y}_i(t)$ has the second meaning.

It is also useful to mention that

$$d(\star, i) - d(\star, j) = \Gamma_i^\star - \Gamma_j^\star$$

so that

$$d(\star, i) \geq d(\star, j),$$

if and only if

$$\Gamma_i^\star \geq \Gamma_j^\star.$$

Theorem 1: For the jth and i^{th} KFs in the bank, under the assumption (1) (ergodicity and stationarity of the residuals) we have

$$\lim_{t \to \infty} k_j^i(Y_t) = 0 \qquad (17)$$

if and only if

$$\Gamma_i^\star \geq \Gamma_j^\star \qquad (18)$$

Proof: Note that

$$\log k_j^i(Y_t) = \sum_{n=1}^{t} \log k_j^i(y(n)|Y_{n-1}). \qquad (19)$$

Under assumption (1) we can compute the expected value of $\log k_j^i(y(n)|Y_{n-1})$ as

$$\lim_{t \to \infty} \frac{1}{t} \sum_{n=1}^{t} \log k_j^i(y(n)|Y_{n-1}) = E\{\log k_j^i(y(n)|Y_{n-1})\}$$
$$= d_n(i;j) = \Gamma_j^\star - \Gamma_i^\star. \qquad (20)$$

If

$$\Gamma_j^\star \leq \Gamma_i^\star \qquad (21)$$

then by comparing (19), (20), and (21), it follows that

$$\lim_{t \to \infty} \log k_j^i(Y_t) = \lim_{t \to \infty} \sum_{n=1}^{t} \log k_j^i(y(n)|Y_{n-1}) = -\infty \qquad (22)$$

which implies that

$$\lim_{t \to \infty} k_j^i(Y_t) = 0. \qquad (23)$$

∎

This theorem shows that the KF which has the minimum $\Gamma_i^\star$ will be selected.

When the open loop plant is stable and in the absence of control action, $\Gamma_i^\star$ is equivalent to the Baram Proximity Measure (BPM), see [4], [10] for more information on the BPM. Based on the results of Theorem 1 we can define a (Pseudo) norm on the set of unstable plants given by

$$m(\theta_i; \theta_j) := |\Gamma_i^\star - \Gamma_j^\star|. \qquad (24)$$

It should be stressed that the above mentioned norm on unstable plants depends on the controller in the feedback loop. In fact, one cannot compute the term $S_i^\star$ in (15) without a stabilizing controller in the loop.

Lemma 1: The defined norm in (24) is a Pseudo Norm.

□

Proof: It is not difficult to see that

$$m(\theta_i; \theta_i) = |\Gamma_i^\star - \Gamma_i^\star| = 0.$$

To prove the symmetry property, use the fact that

$$m(\theta_i; \theta_j) = |\Gamma_i^\star - \Gamma_j^\star| =$$
$$|\Gamma_j^\star - \Gamma_i^\star| = m(\theta_j; \theta_i).$$

The triangle inequality follows from

$$m(\theta_i; \theta_p) + m(\theta_p; \theta_j) =$$
$$|\Gamma_i^\star - \Gamma_p^\star| + |\Gamma_p^\star - \Gamma_j^\star| \geq$$
$$|\Gamma_i^\star - \Gamma_p^\star + \Gamma_p^\star - \Gamma_j^\star| =$$
$$|\Gamma_i^\star - \Gamma_j^\star| = m(\theta_i; \theta_j).$$

∎

The question that should be answered at this stage is how to compute $\Gamma_i^\star$, $i \in \{1, \ldots, N\}$ in (15). The S_i in (15) is computed by solving the Riccati equation corresponding to the ith KF. However, in the computation of $S_i^\star$ the effect of the controller should be considered. Let us assume that the dynamics of the stabilizing controller in the loop are given by

$$x_c(t+1) = A_c x_c(t) + B_c y(t) \qquad (25)$$
$$u(t) = C_c x_c(t).$$

Estimating the states of the plant when the true parameter in the plant is $\theta_\star$ using the ith KF yields

$$\begin{bmatrix} x(t+1) \\ x_c(t+1) \\ \hat{x}_i(t+1) \end{bmatrix} = \begin{bmatrix} A_{\theta_\star} & B_{\theta_\star} C_c & 0 \\ B_c C_{\theta_\star} & A_c & 0 \\ H_{\theta_i} C_{\theta_\star} & B_{\theta_i} C_c & A_{E_i} \end{bmatrix} \begin{bmatrix} x(t) \\ x_c(t) \\ \hat{x}_i(t) \end{bmatrix}$$
$$+ \begin{bmatrix} G_{\theta_\star} & 0 \\ 0 & B_c \\ 0 & H_{\theta_i} \end{bmatrix} \begin{bmatrix} w_t \\ v_t \end{bmatrix}, \qquad (26)$$

where $A_{E_i} = A_{\theta_i} - H_{\theta_i} C_{\theta_i}$. Let

$$A_i^{aug} = \begin{bmatrix} A_{\theta_\star} & B_{\theta_\star} C_c & 0 \\ B_c C_{\theta_\star} & A_c & 0 \\ H_{\theta_i} C_{\theta_\star} & B_{\theta_i} C_c & A_{E_i} \end{bmatrix},$$

$$G_i^{aug} = \begin{bmatrix} G_{\theta_\star} & 0 \\ 0 & B^c \\ 0 & H_{\theta_i} \end{bmatrix}, \quad Q_{aug} = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}.$$

Then, the matrix

$$\Xi_i = \lim_{t \to \infty} E\{[x(t)^T \ x_c(t)^T \ \hat{x}_i(t)^T]^T [x(t) \ x_c(t) \ \hat{x}_i(t)]\},$$

is computed by solving the Lyapunov equation[6]

$$\Xi_i = A_i^{aug} \Xi_i A_i^{augT} + G_i^{aug} Q_{aug} G_i^{augT},$$

from which it follows that

$$\Gamma_i^\star = C_i^{aug} \Xi_i C_i^{augT} + R,$$

where $C_i^{aug} = [C_{\theta_\star} \ 0 \ C_{\theta_i}]$. The existence of $\Gamma_i^\star$ implies the stationarity of the residuals, $\tilde{y}_i(t)$. Moreover, it follows from [12] (pp. 464-467) that the innovation sequence, $\tilde{y}_i(t)$, is ergodic iff

$$\lim_{t \to \infty} \frac{1}{t+1} \sum_{n=0}^{t} |R_i(n)|^2 = 0,$$

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[6]Such a solution always exists since the matrix A_i^{aug} has all its eigenvalues inside the unit circle.
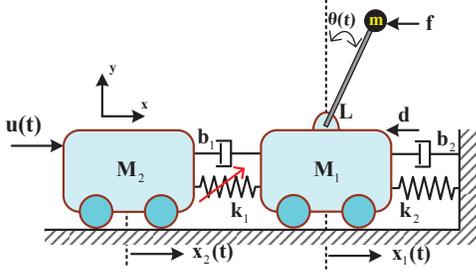
Fig. 2. Inverted pendulum installed on Mass-Spring-Damper (IP-MSD), test-bed example.

where $|R(n)|$ denotes the determinant of the matrix

$$R_i(n) = E\{[y(t) - \hat{y}_i(t)][y(t+n) - \hat{y}_i(t+n)]^T\}$$
$$\begin{cases} C_i^{aug} \Xi_i C_i^{augT} + R, & n = 0 \\ C_i^{aug} \Xi_i C_i^{augT} (A_i^{aug})^n, & n > 0. \end{cases}$$

Since all the eigenvalues of A_i^{aug} are inside the unit circle, we obtain $\|A_i^{aug}\| < 1$ from which it follows that

$$\lim_{t \to \infty} \sum_{n=1}^{t} |R_i(n)|^2 = |C_i^{aug} \Xi_i C_i^{augT}|^2 \lim_{t \to \infty} \sum_{n=1}^{t} |A_i^{aug}|^{2n}$$
$$= \frac{|C_i^{aug} \Xi_i C_i^{augT}|^2}{1 - |A_i^{aug}|^2} < \infty,$$

which implies that

$$\lim_{t \to \infty} \frac{1}{t+1} \sum_{n=0}^{t} |R_i(n)|^2 = 0$$

and hence, it follows that the innovation sequence are ergodic (see from [12] (pp. 464-467)).

IV. ILLUSTRATIVE EXAMPLE

The proposed MMAE procedure is now evaluated through an example of an inverted pendulum installed on the two-cart mass-spring-damper (IP-MSD) depicted in Fig. 2. As shown in Fig. 2, IP-MSD consists of a thin rod attached to a moving cart connected to a wall with a known spring and damper and to another cart through a known damper and a spring with unknown stiffness coefficient. Whereas a normal pendulum is stable when hanging downwards, a vertical inverted pendulum is inherently unstable, and must be actively balanced in order to remain upright, typically by moving the cart horizontally as part of a feedback system. We have designed a robust controller which stabilizes the plant for all the values of the parametric uncertainty set, see [13] for details on designing robust controllers for this plant using the Mixed-μ synthesis [14]. Throughout this section we use a sample time of $Ts = 1$ ms in all the numerical simulations. The system in Fig. 2 includes random colored disturbance forces, $d(t)$ and $f(t)$, acting on mass M_1 and m. The control force $u(t)$ acts upon the mass M_2. The disturbance forces $d(t)$ and $f(t)$ are stationary first-order (colored) independent stochastic process generated by driving a low-pass filter, $W_d(s)$, with independent continuous-time white noises $w_1(t)$

and $w_2(t)$; with zero mean and intensities of $W_1 = 0.1$ and $W_2 = 10^{-5}$,

$$d(s) = \frac{.8}{s + 0.8} w_1(s), \tag{27}$$

$$f(s) = \frac{.8}{s + 0.8} w_2(s). \tag{28}$$

The position of the cart 1, $x_1(t)$, (in meters) and the angle of the pendulum $\theta(t)$ (in radians) are measured outputs that are corrupted by independent zero mean white noise with intensity of 10^{-7} and 10^{-8}, respectively. The following parameters in the equations are fixed and known:

$$M_1 = M_2 = 1 \,(\text{kg}); \quad m = .25 \,(\text{kg}); \quad k_2 = .15 \,(\text{N/m});$$
$$b_1 = b_2 = .1 \,(\text{N s/m}); \quad L = 1 \,(\text{m}). \tag{29}$$

The upper and lower-bounds for the uncertain spring constant, k_1, are:

$$k_1 \in \{k_1 : \ 0.5 \leqslant k_1 \leqslant 2.5\}. \tag{30}$$

Five KFs based on the nominal values of

$$k_1 \in \{0.632, \ 0.94, \ 1.27, \ 1.64, \ 2.177\}. \tag{31}$$

were designed. In the case that the true uncertain parameter is not one of the nominal values selected in (31), theorem 1 tells us that the a posterior probability assigned to the KF whose model is closest to the true plant (in the sense of the above defined norm) converges to one (and clearly the other probabilities converge to zero).

Fig. 3 shows the distance of the true plant (for different values of uncertain parameter) and the nominal models selected by (31). It follows from the Fig. 3 that if the uncertain k_1 lies in $[0.5 \ 0.797]\,(\text{N/m})$, then the probability assigned to the first KF will converge to one, since $m(0.632; k_1)$ is the smallest among $m(0.94; k_1)$, $m(1.27; k_1)$, $m(1.64; k_1)$ and $m(2.177; k_1)$. In other words, the model adopted based on the first nominal value in (31), i.e. $0.632\,(\text{N/m})$, is the closest to the true plant when uncertain k_1 lies in $[0.5 \ 0.797]\,(\text{N/m})$. Similarly, the model adopted based on the second nominal value in (31), $0.94\,(\text{N/m})$, is the closest to the true plant when the uncertain k_1 lies in $[0.797 \ 1.118]\,(\text{N/m})$; similar conclusions apply to the other cases. Fig. 4 shows the results of a simple simulation. In this experiment the spring coefficient k_1 is constant ($k_1 = 1.05\,(\text{N/m})$). Fig. 4 represents the time evolution of the a posterior probabilities in the MMAE and it shows that in about 2 seconds the correct model is identified. Fig. 5 depicts the time evolution of the output (x_1 and θ) and the estimated output ($\hat{x}_1$ and $\hat{\theta}$), respectively.

We stress that the performance of any adaptive system must be evaluated not only for constant unknown parameters but also, for time-varying parameters which undergo slow or rapid time-variations. Fig. 6 shows the results of the simulation where uncertain parameter changes in time. In this experiment k_1 varies according to the sinusoidal waveform shown in the first subplot of Fig. 6. The model boundaries, as defined Fig. 3, are also shown (using the dashed lines) in the first subplot. Note that the a posterior probabilities respond very fast to the model changes. It is shown that the
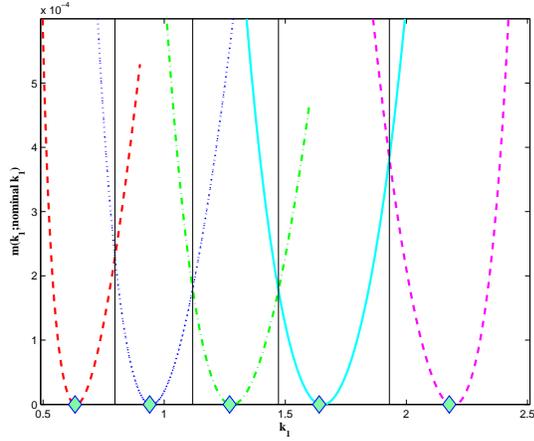
Fig. 3. Nominal values of uncertain parameter and the distance between nominal models and true plant.
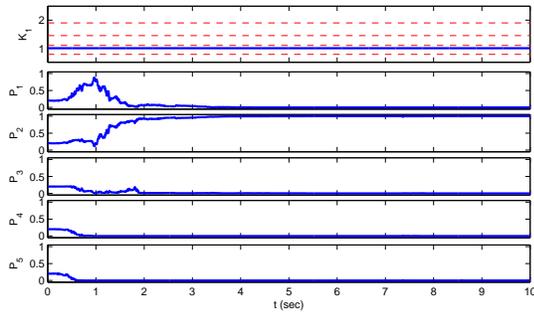
Fig. 4. The a posterior probabilities while $k_1 = 1.05\,(\text{N/m})$

MMAE can trace the changes in the uncertain parameter and adapt accordingly.

V. CONCLUSIONS

This paper studied the application of Multiple Model Adaptive Estimation (MMAE) techniques to open-loop unstable plants. We proved that by having a stabilizing controller in the feedback loop, the residuals of the KFs in the bank are ergodic and stationary; moreover, we showed that under the ergodicity and stationarity of the residuals, the a posterior probability assigned to the KF corresponding to the model that is closest to the true plant converges to one while the others probabilities converge to zero. We also developed a pseudo norm on open-loop unstable dynamic linear systems. The defined norm depends on the dynamics of the stabilizing controller in the feedback loop. The performance and convergence of the MMAE procedure were illustrated with Monte-Carlo simulation runs using the model of an inverted pendulum installed on a system of masses, springs and dampers.

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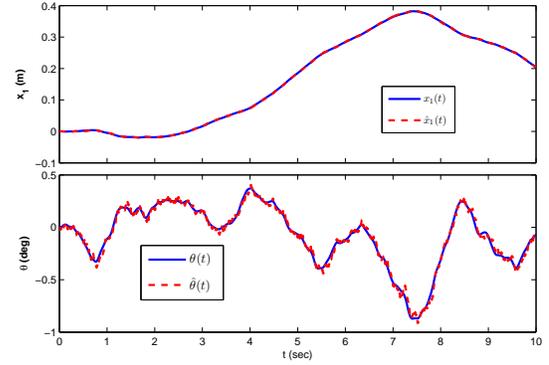
Fig. 5. Output Estimations while $k_1 = 1.05\,(\text{N/m})$
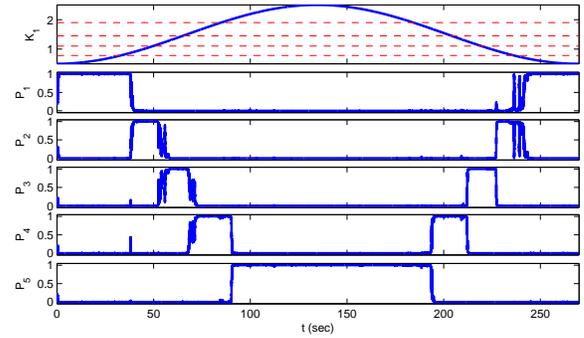
Fig. 6. The a posterior probabilities while k_1 varies in the time.

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