Observability analysis of 2D single beacon navigation in the presence of constant currents for two classes of maneuvers

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Abstract: We analyze the observability properties of an underwater vehicle (moving in 2D) performing single beacon navigation for two specific classes of maneuvers, whereby the vehicle measures its distance to a fixed transponder located at a known position using an acoustic ranging device. We show that in the presence of known ocean currents, the system is found to be globally observable for constant relative course and constant (nonzero) relative course rate inputs in the sense of Herman and Krener. On the other hand, with unknown ocean currents the system fails to be locally weakly observable with constant relative course but we characterize the set of indistinguishable states from a given initial position and ocean current configuration. Interestingly, observability can be achieved with constant (nonzero) relative course rate in the presence of unknown, constant ocean currents.

Keywords: Single beacon navigation, Range-only measurement, Set of indistinguishable states, Observability.

1. INTRODUCTION

In marine robotics, accurate knowledge of the position of a vehicle is necessary for precise navigation. The position of a marine vehicle can be estimated using different sensors suites and methods. In recent years, single beacon navigation using range measurement has received widespread attention because of its potential low cost application but at the present time is still a challenging problem. A rigorous observability analysis of the single beacon navigation using range measurement is necessary before designing a good estimator.

In the literature, different types of models have been studied for single beacon navigation in 2D/3D. Further, the observability issue has been addressed using different approaches such as linearization, (Gadre and Stilwell, 2004)-(Gadre and Stilwell, 2005), geometric methods (Arrichiello et al., 2011) and algebraic methods (Jouffroy and Reger, 2006). In (Gadre and Stilwell, 2004), a nonlinear system with position and heading (assuming negligible sideslip) as the state vector is considered while the linear velocity and heading rate is considered as an input. The output functions are the 2D range and heading. The nonlinear system is linearized about a nominal trajectory and the standard observability results of Linear Time Varying (LTV) are used to analyze the observability properties (Rugh, 1996). In (Gadre and Stilwell, 2005), unknown constant ocean currents are augmented into the state vector and the same procedure of (Gadre and Stilwell, 2004) is applied.

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The authors in (Arrichiello et al., 2011) exploited the nonlinear observability concepts of a nonlinear inter-vehicle ranging system using Herman-Krener observability rank conditions of local weak observability (Hermann and Krener, 1977) and the results obtained are validated experimentally in an equivalent single beacon navigation scenario. Moreover, they compute analytically two unobservability metrics, given by the inverse of the minimum singular value and the ratio between the maximum and minimum singular values of the observability matrix for the proposed system.

In (Jouffroy and Reger, 2006), the authors study the position estimation problem for the 2D kinematic model of an underwater vehicle using a single acoustic transponder with knowledge of the body-fixed velocities and the heading of the vehicle. The heading rate is not included in state, as it can be measured using an IMU. The model is then converted into polar coordinates. Further observability analysis is carried out in the algebraic set-up. In other words, the state is expressed by a function whose arguments are the output of the system and its first derivative in polar coordinates.

In (Parlangeli et al., 2012), the observability of the 3D nonholonomic floating vehicle based on range only measurements is investigated. The problem considered in this paper is the relative localization of two vehicle using the distance between the vehicles as measurement, and linear and angular velocities as inputs. Using analytical approach, sets of indistinguishable states from a given initial position and orientation configuration is computed for two specific inputs, namely, i) both vehicles have zero linear velocity ii) one vehicle with zero linear velocity and the other nonzero linear velocity.

Recently, in (Bayat and Aguiar, 2012) the observability problem of the Simultaneous Localization and Mapping (SLAM) process of an Autonomous Underwater Vehicle (AUV) equipped with inertial sensors, a depth sensor, and an acoustic ranging device that provides relative range measurements to stationary beacons is investigated. For trimming trajectories, it is shown that the set of indistinguishable states from a given initial state with the knowledge of either one of the beacon position or the AUV position, contains only the zero vector with exception of a distinct case where there is an additional isolated point.

For Linear Time Invariant (LTI) systems, observability can be verified using the well-known observability matrix and the kernel of the observability matrix that represents the unobservable subspace (Rugh, 1996). Further, for LTI system it is sufficient to check the observability of the system at the origin and the unobservable linear subspace for any nonzero initial configuration can be obtained from the unobservable subspace of the zero configuration (which is an affine hyperplane). Furthermore, for the LTI case, the input does not affect the observability. More precisely, if the LTI system is observable, then the LTI system is also observable for any admissible input. However, in the context of nonlinear systems, which is the case considered in the paper, the set of indistinguishable states (that are not necessarily a linear subspace) depends on the initial configuration and actuator-sensor configuration of the system.

Note that most of the existing results on the observability of nonlinear systems in the literature only give information about local observability and does not provide any information about the set of indistinguishable states from a given initial configuration, that is, the set of all initial configurations that produce identical output time-histories from a given initial configuration for every admissible input. Further, Herman-Krener rank condition is also a sufficient condition (Hermann and Krener, 1977) for local observability and suffers from the fact that it do not provide any information about set of indistinguishable states of the system from a given initial state when the rank condition fails. On the other hand, when the nonlinear system is locally observable at a given initial state in the sense of Herman-Krener, it means that there exists an input which can distinguish every state in an open neighborhood of the given initial condition from the given initial condition. Notice that, this does not mean that every admissible input is able to do so. Hence, practically, there is a need to identify a class of admissible inputs with the property that every input has the ability to distinguish every pair of initial configurations through the outputs.

In this paper, we analyze the observability properties of an underwater vehicle moving in 2D performing single beacon navigation for two specific classes of maneuvers, whereby the vehicle measures its distance to a fixed transponder located at a known position using an acoustic ranging device. We show that in the presence of known ocean currents, the system is found to be globally observable for constant relative course and constant (nonzero) relative course rate inputs in the sense of Herman and Krener. On the other hand, with unknown ocean currents the system fails to be locally weakly observable with constant relative course but we characterize the set of indistinguishable states from a given initial position and ocean current configuration. Interestingly, observability can be achieved with constant (nonzero) relative course rate in the presence of unknown, constant ocean currents.

The organization of the paper is as follows. In Section 3, we introduce the basic definitions of observability in the context of nonlinear systems. In Section 4, we address the single beacon modeling issue and in the subsequent section, we analyze the observability properties of the single beacon system for two important but simple class of inputs. In Section 6, we conclude our results. We begin with mathematical preliminaries that are used in the rest of the paper.

2. MATHEMATICAL PRELIMINARIES

Given $(a,b) \in \mathbb{R} \times \mathbb{R}$ such that $a^2 + b^2 \neq 0$, we let $\text{atan2}(b,a)$ denote the unique angle $\theta \in [0, 2\pi)$ satisfying $\sin\theta = b/\sqrt{a^2+b^2}$ and $\cos\theta = a/\sqrt{a^2+b^2}$. Given $a, b \in \mathbb{R}$, we write $a = b \mod 2\pi$ if there exists $k \in \mathbb{Z}$ such that $a = b + 2k\pi$. Given $\boldsymbol{x} := [x_1\ x_2]^{\mathrm{T}} \in \mathbb{R}^2 \setminus \{0\}$, define $\Theta(\boldsymbol{x}) \stackrel{\text{def}}{=} \text{atan2}(x_2, x_1)$.

We denote the Euclidean norm in $\mathbb{R}^2$ by $\|\cdot\|$ and the determinant of a matrix $A \in \mathbb{R}^{n\times n}$ by $\det(A)$. Given $\phi \in [0, 2\pi)$, we define the orthonormal vectors $\breve{\boldsymbol{u}}(\phi) \stackrel{\text{def}}{=} [\cos\phi\ \sin\phi]^{\mathrm{T}} \in \mathbb{R}^2$ and $\breve{\boldsymbol{u}}^{\perp}(\phi) \stackrel{\text{def}}{=} [-\sin\phi\ \cos\phi]^{\mathrm{T}} \in \mathbb{R}^2$, that is, $\|\breve{\boldsymbol{u}}(\phi)\| = \|\breve{\boldsymbol{u}}^{\perp}(\phi)\| = 1$ and $\breve{\boldsymbol{u}}(\phi)^{\mathrm{T}}\breve{\boldsymbol{u}}^{\perp}(\phi) = 0$. Given $\beta_1, \beta_2 \in [0, 2\pi)$, it is easy to show that $\breve{\boldsymbol{u}}(\beta_1)^{\mathrm{T}}\breve{\boldsymbol{u}}(\beta_2) = \cos(\beta_1 - \beta_2)$ and $\breve{\boldsymbol{u}}(\beta_1)^{\mathrm{T}}\breve{\boldsymbol{u}}^{\perp}(\beta_2) = \sin(\beta_1 - \beta_2)$. We denote the set of all k times differentiable functions defined between X and Y by $\mathrm{C}^k(X;Y)$.

3. OBSERVABILITY OF NONLINEAR SYSTEMS

Consider the nonlinear control system

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}), \\ \boldsymbol{y} &= \boldsymbol{h}(\boldsymbol{x}), \end{aligned} \tag{3.1}$$

where $\boldsymbol{F}$ is a complete and smooth vector field on $\mathbb{R}^n$, the input vector $\boldsymbol{u}$ takes values in a compact subset Ω of $\mathbb{R}^p$ containing zero in its interior, and the output function $\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^q$ has smooth components. We restrict $\boldsymbol{u}$ to be piecewise constant control inputs and for each $\boldsymbol{u} \in \Omega$, $\boldsymbol{F}_{\boldsymbol{u}} := \boldsymbol{F}(\cdot, \boldsymbol{u})$ is a smooth vector field. Define $\mathcal{D} := \{\boldsymbol{F}_{\boldsymbol{u}} \colon \boldsymbol{u} \in \Omega\}$. We recall the following definitions from (Hermann and Krener, 1977; Nijmeijer and der Schaft, 1990).

Definition 3.1. Two states $\boldsymbol{z}, \boldsymbol{z}' \in \mathbb{R}^n$ are *indistinguishable* for the system (3.1) on $[0, t_f]$ if, for every admissible input $\boldsymbol{u}$, the solutions of (3.1) satisfying the initial conditions $\boldsymbol{x}(0) = \boldsymbol{z}$ and $\boldsymbol{x}(0) = \boldsymbol{z}'$ produce identical output-time histories on $[0, t_f]$.

For every $\boldsymbol{z} \in \mathbb{R}^n$, let $\mathcal{I}(\boldsymbol{z}) \subseteq \mathbb{R}^n$ denote the set of all states that are indistinguishable from $\boldsymbol{z}$. Note that indistinguishibility is an equivalence relation.

Definition 3.2. The system (3.1) is *observable at* $\boldsymbol{z} \in \mathbb{R}^n$ if $\mathcal{I}(\boldsymbol{z}) = \{\boldsymbol{z}\}$, and is *observable* if $\mathcal{I}(\boldsymbol{z}) = \{\boldsymbol{z}\}$ for every $\boldsymbol{z} \in \mathbb{R}^n$.

Definition 3.3. The system (3.1) is *locally weakly observable at* $\boldsymbol{z} \in \mathbb{R}^n$ if $\boldsymbol{z}$ is an isolated point of $\mathcal{I}(\boldsymbol{z})$ and is *locally weakly observable* if it is locally weakly observable at every $\boldsymbol{z} \in \mathbb{R}^n$.

Note that observability (O) implies local weak observability (LWO).

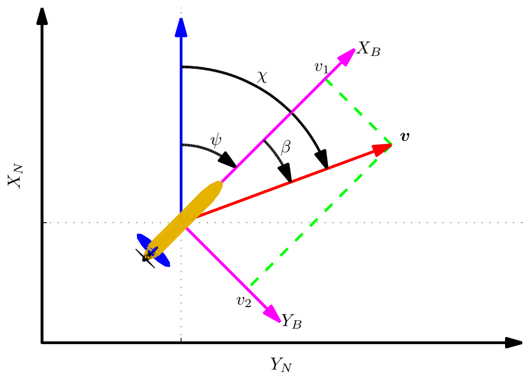

Fig. 1. Geometrical representation of course, heading and sideslip angles

4. 2D SINGLE BEACON MODEL

In this section we describe the 2D single beacon model considered in the paper.

Definition 4.1. (Relative course angle χ). In the absence of currents, this is the usual course angle defined from the x-axis of the North-East-Down (NED) frame to the velocity vector v of the vehicle, positive rotation about the z-axis of the NED frame using the right-hand screw convention. In the presence of currents, the angle is defined with respect to the relative velocity of the vehicle with respect to the water, that is, the total inertial velocity minus the current velocity.

Definition 4.2. (Heading (yaw) angle ψ). The angle from the NED x-axis to the body x-axis, positive rotation about the z-axis of the NED frame by the right-hand screw convention.

Definition 4.3. (Sideslip (drift) angle β). The angle from the body x-axis to the relative velocity vector of the vehicle, positive rotation about the body z-axis frame by the right-hand screw convention.

Note that from the geometry of figure (1), it is clear that $\chi = \psi + \beta$ and the sideslip angle $\beta = \sin^{-1}(v_2/\|\boldsymbol{v}\|)$. Further, for small sideslip $\beta \approx v_2/\|\boldsymbol{v}\|$. In fact, when the sway velocity $v_2 = 0$, the heading angle equals the course angle, that is, there is no sideslip.

The 2D kinematic model of a marine vehicle measuring the distance to a single transponder located at a known position vector $\boldsymbol{b} \in \mathbb{R}^2$ is given by

$$\dot{\boldsymbol{x}}(t) = v(t)\,\breve{\boldsymbol{u}}(\chi(t)) + \boldsymbol{v}_{\rm c}(t), \tag{4.1}$$

$$\dot{\boldsymbol{v}}_{\rm c}(t) = \boldsymbol{0}, \tag{4.2}$$

$$y(t) = \|\boldsymbol{x}(t) - \boldsymbol{b}\|, \tag{4.3}$$

where $t \in J \overset{\text{def}}{=} [0, t_f] \subset \mathbb{R}$, $K \overset{\text{def}}{=} [0, 2\pi)$, $\boldsymbol{x}(t) \in \mathbb{R}^2 \setminus \{\boldsymbol{b}\}$ is the instantaneous inertial position vector, $\boldsymbol{v}_{\rm c}(t) \in \mathbb{R}^2$ is a disturbance ocean (constant) current vector, $\chi \in \mathcal{U} \subseteq \mathrm{C}^k(J; K)$, $k \geq 0$, is the known relative course angle input, and $v \in \mathcal{V} \subseteq \mathrm{C}^l(J; \mathbb{R}_+)$, $l \geq 0$, is the linear velocity of the vehicle in the body-frame.

Assumption 4.4. Without loss of generality, we assume that the beacon is at the origin.

Note that, if $\boldsymbol{b} \in \mathbb{R}^2 \setminus \{\boldsymbol{0}\}$, we can define a transformation $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{v}}_{\rm c}) \overset{\text{def}}{=} (\boldsymbol{x} - \boldsymbol{b}, \boldsymbol{v}_{\rm c})$. Then, $(\dot{\tilde{\boldsymbol{x}}}, \dot{\tilde{\boldsymbol{v}}}_{\rm c}) = (\dot{\boldsymbol{x}}, \dot{\boldsymbol{v}}_{\rm c})$ and $\tilde{y} = \|\tilde{\boldsymbol{x}}\|$. In the new variable $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{v}}_{\rm c})$, the beacon is at the origin.

5. OBSERVABILITY ANALYSIS OF SINGLE BEACON NAVIGATION

Equations (4.1)-(4.3) define a nonlinear input-affine system with state $(\boldsymbol{x}, \boldsymbol{v}_{\rm c}) \in \mathcal{M} \overset{\text{def}}{=} \mathbb{R}^2 \setminus \{\boldsymbol{0}\} \times \mathbb{R}^2$, drift vector field $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{v}_{\rm c}) = (\boldsymbol{v}_{\rm c}, \boldsymbol{0})$, control vector field $\boldsymbol{G}_1(\boldsymbol{x}, \boldsymbol{v}_{\rm c}) = (v\,\breve{\boldsymbol{u}}(\psi), \boldsymbol{0})$, and the output function $h(\boldsymbol{x}, \boldsymbol{v}_{\rm c}) = \|\boldsymbol{x}\|$. The solution for the initial state $\mathbf{x}_0 := (\boldsymbol{x}_0, \boldsymbol{v}_{{\rm c}_0}) \in \mathcal{M}$ at time $t \in J$ with input $\boldsymbol{u} := (v, \psi) \in \mathcal{U} \times \mathcal{V}$ is denoted by $\Phi_t^{\boldsymbol{u}}(\mathbf{x}_0)$ and is given by

$$\Phi_t^{\boldsymbol{u}}(\mathbf{x}_0) = \mathbf{x}_0 + \begin{bmatrix} \boldsymbol{v}_{{\rm c}_0}\, t + \displaystyle\int_0^t v(\tau)\,\breve{\boldsymbol{u}}(\chi(\tau))\,\mathrm{d}\tau \\ \boldsymbol{0} \end{bmatrix},$$

where

$$\int_0^t v(\tau)\,\breve{\boldsymbol{u}}(\chi(\tau))\,\mathrm{d}\tau \overset{\text{def}}{=} \begin{bmatrix} \displaystyle\int_0^t v(\tau)\cos\chi(\tau)\,\mathrm{d}\tau \\ \displaystyle\int_0^t v(\tau)\sin\chi(\tau)\,\mathrm{d}\tau \end{bmatrix},$$

while the output is given by

$$h(\Phi_t^{\boldsymbol{u}}(\mathbf{x}_0)) = \left\| \boldsymbol{x}_0 + \boldsymbol{v}_{{\rm c}_0}\, t + \int_0^t v(\tau)\,\breve{\boldsymbol{u}}(\chi(\tau))\,\mathrm{d}\tau \right\|.$$

At this point, we will make use of the following result in order to simplify the observability analysis. The result essentially means that the observability properties of the system (4.1)-(4.2) with range squared measurement and range measurement are equivalent.

Lemma 5.1. The system (4.1)-(4.2) with range squared measurement is GO (respectively, LWO) if and only if the system (4.1)-(4.2) with range only measurement is GO (respectively, LWO).

Proof. First, suppose the system (4.1)-(4.2) with range squared is observable. Then, for every distinct pair of initial conditions $\mathbf{x}_0, \mathbf{z}_0 \in \mathcal{M}$, there exists $t^* \in J$ and an input $\boldsymbol{u}^* \in \mathcal{U} \times \mathcal{V}$ such that $h(\Phi_{t^*}^{\boldsymbol{u}^*}(\mathbf{x}_0))^2 \neq h(\Phi_{t^*}^{\boldsymbol{u}^*}(\mathbf{z}_0))^2$ or, equivalently, $h(\Phi_{t^*}^{\boldsymbol{u}^*}(\mathbf{x}_0)) \neq h(\Phi_{t^*}^{\boldsymbol{u}^*}(\mathbf{z}_0))$, which means that the system (4.1)-(4.2) with range is observable. The converse follows similarly. This proves the claim.

In this paper we study the observability properties of model (4.1)-(4.3) for three distinct cases, in the presence of two classes of inputs. The three cases are: *i)* No ocean current, *ii)* Known ocean current, *iii)* Unknown ocean current. The two classes of inputs are constant relative course and nonzero constant relative course rate, that is,

i) $\mathcal{U}_{\rm con} \overset{\text{def}}{=} \{\chi(t) = \phi,\ \forall\, t \in J :\ \phi \in K\}$,

ii) $\mathcal{U}_{\rm cir} \overset{\text{def}}{=} \{\chi(t) = \omega\, t + \phi,\ \forall\, t \in J \colon \omega > 0,\ \phi \in K\}$.

In the following sections, we characterize set of indistinguishable states for cases $(i) - (iii)$ subject to the above two classes of inputs.

5.1 NO OCEAN CURRENTS

The system without ocean currents is described by

$$\dot{\boldsymbol{x}} = v\,\breve{\boldsymbol{u}}(\chi), \tag{5.1}$$

$$y = \|\boldsymbol{x}\|, \tag{5.2}$$

with state $\boldsymbol{x} \in \mathcal{M} := \mathbb{R}^2 \setminus \{0\}$ and input $\boldsymbol{u} = (v, \chi) \in \mathcal{U} \times \mathcal{V}$. Let $\mathcal{I}_1(\boldsymbol{x}_0)$ be the set of indistinguishable states from a given state $\boldsymbol{x}_0 \in \mathcal{M}$ associated with system (5.1)-(5.2). Note that in this case χ and χ are the same.

Assumption 5.2. Without loss of generality, we assume that $v > 0$ is a constant.

The reason for assuming constant v is as follows. Re-write (5.1) as follows

$$\frac{1}{v}\,\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \breve{\boldsymbol{u}}(\chi). \tag{5.3}$$

Pick $\delta > 0$ and define a transformation $\delta\,\dot{\tau}(t) = v(t)$. Then, (5.3) becomes

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\,\frac{\mathrm{d}t}{\mathrm{d}\tau} = \delta\,\breve{\boldsymbol{u}}(\chi). \tag{5.4}$$

We can assume $\delta = 1$. Without loss of generality, we assume that v is constant.

Suppose $v \equiv 0$. Then, for every $\boldsymbol{x}_0 \in \mathcal{M}$, $h(\Phi_t^{\boldsymbol{u}}(\boldsymbol{x}_0)) = \|\boldsymbol{x}_0\|$ for all $t \in J$ which is a constant. Hence, for $v = 0$, it follows that $\mathcal{I}_1(\boldsymbol{x}_0) = \{\boldsymbol{z} \in \mathcal{M} :\, \|\boldsymbol{z}\|^2 = \|\boldsymbol{x}_0\|^2\}$. Hence we assume that $v > 0$.

Constant relative course The following result characterizes the set of indistinguishable states for the system (5.1)-(5.2) with $\mathcal{U}_{\text{con}}$.

Proposition 5.3. Consider the system (5.1)-(5.2) and let $\mathcal{U}_{\text{con}}$ be the set of admissible inputs. Then, $\mathcal{I}_1(\boldsymbol{x}_0) = \{\boldsymbol{x}_0\}$ for every $\boldsymbol{x}_0 \in \mathcal{M}$.

Proof. Consider $\boldsymbol{x}_0 \in \mathcal{M}$ and let $\boldsymbol{z} \in \mathcal{I}_1(\boldsymbol{x}_0)$. Then, by the definition $h(\Phi_t^{\boldsymbol{u}}(\boldsymbol{z})) = h(\Phi_t^{\boldsymbol{u}}(\boldsymbol{x}_0))$ for every $t \in J$, $\chi \in \mathcal{U}_{\text{con}}$ and $v > 0$. This implies that $\|\boldsymbol{z} + v\,\breve{\boldsymbol{u}}(\chi)\,t\|^2 = \|\boldsymbol{x}_0 + v\,\breve{\boldsymbol{u}}(\chi)\,t\|^2$, that is, $\|\boldsymbol{z}\|^2 + 2\,v\,t\,\boldsymbol{z}^{\mathrm{T}}\breve{\boldsymbol{u}}(\chi) = \|\boldsymbol{x}_0\|^2 + 2\,v\,t\,\boldsymbol{x}_0^{\mathrm{T}}\breve{\boldsymbol{u}}(\chi)$ for every $t \in J$, $\chi \in \mathcal{U}_{\text{con}}$ and $v > 0$. This is true only if $\|\boldsymbol{z}\|^2 = \|\boldsymbol{x}_0\|^2$ and $\boldsymbol{z}^{\mathrm{T}}\breve{\boldsymbol{u}}(\chi) = \boldsymbol{x}_0^{\mathrm{T}}\breve{\boldsymbol{u}}(\chi)$ for every $\chi \in \mathcal{U}_{\text{con}}$.

Since $\boldsymbol{z}^{\mathrm{T}}\breve{\boldsymbol{u}}(\chi) = \boldsymbol{x}_0^{\mathrm{T}}\breve{\boldsymbol{u}}(\chi)$ for every $\chi \in \mathcal{U}_{\text{con}}$, we conclude that $\boldsymbol{z} = \boldsymbol{x}_0$. Consequently, $\mathcal{I}_1(\boldsymbol{x}_0) \subseteq \{\boldsymbol{x}_0\}$. The reverse inclusion is trivial. This completes the proof.

Remark 5.4. Note that for a given initial condition $\boldsymbol{x}_0 \in \mathcal{M}$, every input $\chi \in \mathcal{U}_{\text{con}}$ does not distinguish every other point $\boldsymbol{z} \in \mathcal{M}$ from $\boldsymbol{x}_0$. Further, the unique input $\chi = \Theta(\boldsymbol{x}_0) \in \mathcal{U}_{\text{con}}$ distinguishes every other point $\boldsymbol{z} \in \mathcal{M}$ from the given point $\boldsymbol{x}_0$.

Remark 5.5. Given an initial condition $\boldsymbol{x}_0 \in \mathcal{M}$, for every input $\chi \neq \Theta(\boldsymbol{x}_0) \in \mathcal{U}_{\text{con}}$, the initial condition $\boldsymbol{z}_0 = \|\boldsymbol{x}_0\|\,\boldsymbol{u}(2\chi - \Theta(\boldsymbol{x}_0))$ is indistinguishable from the given point $\boldsymbol{x}_0$.

Constant (nonzero) relative course rate The solution of the system (5.1)-(5.2) for the initial condition $\boldsymbol{x}_0 \in \mathcal{M}$ and inputs $\chi(t) = \omega\,t + \phi \in \mathcal{U}_{\text{cir}}$, $\omega > 0$, $\phi \in K$, $v > 0$, at time $t \in J$ is given by

$$\Phi_t^{\boldsymbol{u}}(\boldsymbol{x}_0) = \boldsymbol{x}_0 + v\,\omega^{-1}\,[\breve{\boldsymbol{u}}^{\perp}(\phi) - \breve{\boldsymbol{u}}^{\perp}(\omega t + \phi)].$$

From the above equation, it immediately follows that

$$\|\Phi_t^{\boldsymbol{u}}(\boldsymbol{x}_0) - \boldsymbol{x}_{\mathbf{c}}\|^2 = R^2,\ \forall\, t \in J,$$

where $\boldsymbol{x}_{\mathbf{c}} \stackrel{\text{def}}{=} \boldsymbol{x}_0 + v\,\omega^{-1}\,\breve{\boldsymbol{u}}^{\perp}(\phi)$ and $R \stackrel{\text{def}}{=} v\,\omega^{-1}$, which is an equation of a circle with center $\boldsymbol{x}_{\mathbf{c}} \in \mathbb{R}^2$ and radius $R > 0$. In particular, $\boldsymbol{x}_{\mathbf{c}} = 0$ represents a circular motion about the beacon with radius $v\,\omega^{-1}$. The following result characterizes set of indistinguishable states for the system (5.1)-(5.2) subject to input class $\mathcal{U}_{\text{cir}}$.

Proposition 5.6. Consider the system (5.1)-(5.2) and let $\mathcal{U}_{\text{cir}}$ be the set of admissible inputs. Then, $\mathcal{I}_1(\boldsymbol{x}_0) = \{\boldsymbol{x}_0\}$ for every $\boldsymbol{x}_0 \in \mathcal{M}$.

Proof. Consider $\boldsymbol{x}_0 \in \mathcal{M}$ and let $\boldsymbol{z} \in \mathcal{I}_1(\boldsymbol{x}_0)$. Then $h(\Phi_t^{\boldsymbol{u}}(\boldsymbol{z})) = h(\Phi_t^{\boldsymbol{u}}(\boldsymbol{x}_0))$ for all $t \in J$, $\chi \in \mathcal{U}_{\text{cir}}$, $v > 0$, implies that $\|\boldsymbol{z} + \boldsymbol{c}(t)\|^2 = r^2(t)$, where

$$\boldsymbol{c}(t) \stackrel{\text{def}}{=} v\,\omega^{-1}[\breve{\boldsymbol{u}}^{\perp}(\phi) - \breve{\boldsymbol{u}}^{\perp}(\omega t + \phi)],$$
$$r^2(t) \stackrel{\text{def}}{=} \|\boldsymbol{x}_0 + \boldsymbol{c}(t)\|^2.$$

Define $\mathcal{B} \stackrel{\text{def}}{=} \bigcap_{t \geq 0}\left\{\boldsymbol{z}_0 \in \mathcal{M} :\, \|\boldsymbol{z}_0 + \boldsymbol{c}(t)\|^2 = r^2(t)\right\}$. Clearly, $\boldsymbol{x}_0 \in \mathcal{B}$ and hence $\mathcal{B} \neq \emptyset$. We claim that $\mathcal{B} = \{\boldsymbol{x}_0\}$. To show the claim, assume that $\mathcal{B} \neq \{\boldsymbol{x}_0\}$ and consider $\boldsymbol{z}_0 \in \mathcal{B}$ such that $\boldsymbol{z}_0 \neq \boldsymbol{x}_0$. Then, for all $t \in J$, $\boldsymbol{z}_0$ satisfies $\|\boldsymbol{z}_0 + \boldsymbol{c}(t)\|^2 = r^2(t)$. In particular, at time $t = 0$, we have $\|\boldsymbol{z}_0\|^2 = \|\boldsymbol{x}_0\|^2$. Further evaluating at $t = 0.5\,\omega^{-1}\,\pi$ and $t = \omega^{-1}\,\pi$, we have

$$\left(\|\boldsymbol{z}_0\|^2 - \|\boldsymbol{x}_0\|^2\right) + 2\,v\,\omega^{-1}(\boldsymbol{z}_0 - \boldsymbol{x}_0)^{\mathrm{T}}\breve{\boldsymbol{u}}(\phi) + \breve{\boldsymbol{u}}^{\perp}(\phi) = 0,$$
$$\left(\|\boldsymbol{z}_0\|^2 - \|\boldsymbol{x}_0\|^2\right) + 4\,v\,\omega^{-1}(\boldsymbol{z}_0 - \boldsymbol{x}_0)^{\mathrm{T}}\breve{\boldsymbol{u}}^{\perp}(\phi) = 0.$$

Using $\|\boldsymbol{z}_0\|^2 = \|\boldsymbol{x}_0\|^2$ the above two equations can be written in matrix form $A\,\boldsymbol{z}_0 = A\,\boldsymbol{x}_0$, where

$$A = \begin{bmatrix} \cos\phi - \sin\phi & \cos\phi + \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix},$$

and it is easy to verify that $\det(A) = +1$. Hence $\boldsymbol{z}_0 = \boldsymbol{x}_0$, which contradicts our assumption that $\boldsymbol{z}_0 \neq \boldsymbol{x}_0$. Hence $\mathcal{B} = \{\boldsymbol{x}_0\}$.

Thus we have shown that $\boldsymbol{z}_0 \in \mathcal{I}_1(\boldsymbol{x}_0)$ implies $\boldsymbol{z}_0 \in \mathcal{B} = \{\boldsymbol{x}_0\}$. From this it immediately follows that $\mathcal{I}_1(\boldsymbol{x}_0) = \{\boldsymbol{x}_0\}$. This completes the proof.

Remark 5.7. Note that, for every initial state $\boldsymbol{x}_0 \in \mathcal{M}$, $\|\boldsymbol{x}(t)\|^2 = \|\boldsymbol{x}_0\|^2$ for every $t \in J$ if and only if $\boldsymbol{x}_0 = v\,\omega^{-1}$ and $\phi - \Theta(\boldsymbol{x}_0) = n\,\pi + 0.5\,(-1)^n\,\pi$, $n \in \mathbb{Z}$. Notice also that under this condition the motion is a circular motion around the beacon.

The following corollary follows from propositions 5.3 and 5.6.

Corollary 5.8. Consider $\boldsymbol{x}_0 \in \mathcal{M}$. Then (4.1)-(4.2) is observable with respect to $\mathcal{U}_{\text{con}}$ and $\mathcal{U}_{\text{cir}}$.

5.2 KNOWN OCEAN CURRENTS

Consider the single beacon with known current

$$\dot{\boldsymbol{x}}(t) = v\,\breve{\boldsymbol{u}}(\chi(t)) + \boldsymbol{v}_{\text{c}}, \tag{5.5}$$
$$y(t) = \|\boldsymbol{x}(t)\|, \tag{5.6}$$

with state $\boldsymbol{x} \in \mathcal{M} := \mathbb{R}^2\backslash\{0\}$ and input $\boldsymbol{u} = (v, \chi) \in \mathcal{U}\times\mathcal{V}$. Let $\mathcal{I}_2(\boldsymbol{x}_0)$ be the set of indistinguishable states from a given state $\boldsymbol{x}_0 \in \mathcal{M}$ associated with system (5.5)-(5.6). If $\boldsymbol{v}_{\mathbf{c}} = 0$, then (5.5)-(5.6) is same as that discussed in Section 5.1. Hence we assume that $\boldsymbol{v}_{\mathbf{c}} \in \mathbb{R}^2 \setminus \{0\}$. Since $\boldsymbol{v}_{\mathbf{c}}$ is known, we consider it as an additional input to the system. For simplicity we define $\boldsymbol{v}_{\mathbf{t}}(t) \stackrel{\text{def}}{=} \boldsymbol{v}_{\mathbf{c}} + v\,\breve{\boldsymbol{u}}(\chi(t))$.

Suppose $v \equiv 0$. Then $h(\Phi_t^{\boldsymbol{u}}(\boldsymbol{x}_0)) = \|\boldsymbol{x}_0 + \boldsymbol{v}_{\mathbf{c}}\,t\|$. Hence it follows that $\mathcal{I}_1(\boldsymbol{x}_0) = \{\boldsymbol{x}_0\}$. We therefore make the following assumption.

Assumption 5.9. We assume that $v > 0$ is constant.

Constant relative course The solution of (5.5) with initial condition $x_0 \in \mathcal{M}$ for the inputs $\chi \in \mathcal{U}_{\text{con}}$ and $v > 0$ is given by $\Phi_t^{\boldsymbol{u}}(\boldsymbol{x}_0) = \boldsymbol{x}_0 + \boldsymbol{v}_{\mathbf{t}}\,t$, while the output is given by $h(\Phi_t^{\boldsymbol{u}}(\boldsymbol{x}_0)) = \|\boldsymbol{x}_0 + \boldsymbol{v}_{\mathbf{t}}\,t\|$. Note that if $\boldsymbol{v}_{\mathbf{t}} \equiv 0$, then for every $t \in J$, $\Phi_t^{\boldsymbol{u}}(\boldsymbol{x}_0) = \boldsymbol{x}_0$ and $y(t) = \|\boldsymbol{x}_0\|$ which is a constant. It now follows that, if $\boldsymbol{v}_\mathbf{t} \equiv 0$, then $\mathcal{I}_2(\boldsymbol{x}_0) = \{\boldsymbol{z} \in \mathcal{M} : \|\boldsymbol{z}\| = \|\boldsymbol{x}_0\|\}$ for every $\boldsymbol{x}_0 \in \mathcal{M}$. Hence we assume that $\boldsymbol{v}_\mathbf{t} \neq 0$.

Proposition 5.10. Consider the system (5.5)-(5.6) and let $\mathcal{U}_{\mathrm{con}}$ be the class of admissible inputs. Then, $\mathcal{I}_2(\boldsymbol{x}_0) = \{\boldsymbol{x}_0\}$ for every $\boldsymbol{x}_0 \in \mathcal{M}$.

Proof. Consider $\boldsymbol{x}_0 \in \mathcal{M}$ and let $\boldsymbol{z}_0 \in \mathcal{I}_2(\boldsymbol{x}_0)$. Then $h(\Phi_t^{\boldsymbol{u}}(\boldsymbol{z}_0)) = h(\Phi_t^{\boldsymbol{u}}(\boldsymbol{x}_0))$ for every $t \in J$ and $\chi \in \mathcal{U}_{\mathrm{con}}$, $v > 0$. This implies that $\|\boldsymbol{z}_0 + \boldsymbol{v}_\mathbf{t}\, t\| = \|\boldsymbol{x}_0 + \boldsymbol{v}_\mathbf{t}\, t\|$ for every $t \in J$ and $\phi \in \mathcal{U}_{\mathrm{con}}$, where $\boldsymbol{v}_\mathbf{t} = \boldsymbol{v}_\mathbf{c} + v\,\breve{\boldsymbol{u}}(\phi)$. This further implies $(\|\boldsymbol{z}_0\|^2 - \|\boldsymbol{x}_0\|^2) + t\,(\boldsymbol{z}_0 - \boldsymbol{x}_0)^{\mathrm{T}}\boldsymbol{v}_\mathbf{t} = 0$ for every $t \in J$ and $\phi \in \mathcal{U}_{\mathrm{con}}$, which is true only if $\|\boldsymbol{z}_0\|^2 = \|\boldsymbol{x}_0\|^2$ and $(\boldsymbol{z}_0 - \boldsymbol{x}_0)^{\mathrm{T}}\boldsymbol{v}_\mathbf{t} = 0$.

Since $(\boldsymbol{z}_0 - \boldsymbol{x}_0)^{\mathrm{T}}\boldsymbol{v}_\mathbf{t} = 0$ for every $\phi \in \mathcal{U}_{\mathrm{con}}$, we conclude $\boldsymbol{z}_0 = \boldsymbol{x}_0$. Consequently, $\mathcal{I}_2(\boldsymbol{x}_0) \subseteq \{\boldsymbol{x}_0\}$. This completes the proof.

Remark 5.11. Note that for a given initial condition $\boldsymbol{x}_0 \in \mathcal{M}$, every input $\chi \in \mathcal{U}_{\mathrm{con}}$ does not distinguish every other point $\boldsymbol{z} \in \mathcal{M}$ from $\boldsymbol{x}_0$. Further, the unique input $\{\chi \in \mathcal{U}_{\mathrm{con}} \colon \Theta(\boldsymbol{v}_\mathbf{t}) = \Theta(\boldsymbol{x}_0)\}$ distinguishes every other point $\boldsymbol{z} \in \mathcal{M}$ from the given point $\boldsymbol{x}_0$.

Remark 5.12. Given an initial condition $\boldsymbol{x}_0 \in \mathcal{M}$, for every input in the set $\{\chi \in \mathcal{U}_{\mathrm{con}} \colon \Theta(\boldsymbol{v}_\mathbf{t}) \neq \Theta(\boldsymbol{x}_0)\}$ the initial condition $\boldsymbol{z}_0 = \|\boldsymbol{x}_0\|\,\boldsymbol{u}(2\Theta(\boldsymbol{v}_\mathbf{t}) - \Theta(\boldsymbol{x}_0))$ is indistinguishable from the given point $\boldsymbol{x}_0$.

Constant (nonzero) relative course rate The solution of (5.5)-(5.6) with initial condition $\boldsymbol{x}_0 \in \mathcal{M}$ for inputs $\chi(t) := \omega\, t + \phi \in \mathcal{U}_{\mathrm{cir}}$ and $v > 0$ is given by

$$\Phi_t^{\boldsymbol{u}}(\boldsymbol{x}_0) = \boldsymbol{x}_0 + v\,\omega^{-1}\left(\breve{\boldsymbol{u}}^{\perp}(\phi) - \breve{\boldsymbol{u}}^{\perp}(\omega\, t + \phi)\right) + \boldsymbol{v}_\mathbf{c}\, t.$$

We have the following result.

Proposition 5.13. Consider the system (5.5)-(5.6) and let $\mathcal{U}_{\mathrm{cir}}$ be the class of admissible inputs. Then, $\mathcal{I}_2(\boldsymbol{x}_0) = \{\boldsymbol{x}_0\}$ for every $\boldsymbol{x}_0 \in \mathcal{M}$.

Proof. Consider $\boldsymbol{z}_0 \in \mathcal{I}(\boldsymbol{x}_0)$. Then $h(\Phi_t^{\boldsymbol{u}}(\boldsymbol{z}_0)) = h(\Phi_t^{\boldsymbol{u}}(\boldsymbol{x}_0))$ for every $t \in J$ and $\chi \in \mathcal{U}_{\mathrm{cir}}$ and $v > 0$. At $t = 0$, we have $\|\boldsymbol{z}_0\|^2 = \|\boldsymbol{x}_0\|^2$ which implies $\boldsymbol{z}_0 = \|\boldsymbol{x}_0\|\,\breve{\boldsymbol{u}}(\alpha)$, $\alpha \in K$. At $t = 0.5\,\pi\,\omega^{-1}$ and $t = \pi\,\omega^{-1}$, we have

$$\boldsymbol{v}_\mathbf{c}^{\mathrm{T}}[\breve{\boldsymbol{u}}(\alpha) - \breve{\boldsymbol{u}}(\Theta(\boldsymbol{x}_0))] = \bar{k}\,[\breve{\boldsymbol{u}}(\Theta(\boldsymbol{x}_0)) - \breve{\boldsymbol{u}}(\alpha)]^{\mathrm{T}}[\breve{\boldsymbol{u}}(\phi) + \breve{\boldsymbol{u}}^{\perp}(\phi)],$$
$$\boldsymbol{v}_\mathbf{c}^{\mathrm{T}}[\breve{\boldsymbol{u}}(\alpha) - \breve{\boldsymbol{u}}(\Theta(\boldsymbol{x}_0))] = \bar{k}\,[\breve{\boldsymbol{u}}(\Theta(\boldsymbol{x}_0)) - \breve{\boldsymbol{u}}(\alpha)]^{\mathrm{T}}\breve{\boldsymbol{u}}^{\perp}(\phi),$$

where $\bar{k} := 2\,\pi^{-1}\,v$. Equating the right-hand-side of the above two equations yields

$$[\breve{\boldsymbol{u}}(\alpha) - \breve{\boldsymbol{u}}(\Theta(\boldsymbol{x}_0))]^{\mathrm{T}}[\breve{\boldsymbol{u}}(\phi) + \breve{\boldsymbol{u}}^{\perp}(\phi)] = [\breve{\boldsymbol{u}}(\alpha) - \breve{\boldsymbol{u}}(\Theta(\boldsymbol{x}_0))]^{\mathrm{T}}\breve{\boldsymbol{u}}^{\perp}(\phi).$$

This can be simplified as $[\breve{\boldsymbol{u}}(\alpha) - \breve{\boldsymbol{u}}(\Theta(\boldsymbol{x}_0))]^{\mathrm{T}}\breve{\boldsymbol{u}}(\phi) = 0$. Further note that $\breve{\boldsymbol{u}}(\alpha)^{\mathrm{T}}\breve{\boldsymbol{u}}(\phi) = \cos(\alpha - \phi)$ and $\breve{\boldsymbol{u}}(\Theta(\boldsymbol{x}_0))^{\mathrm{T}}\breve{\boldsymbol{u}}(\phi) = \cos(\Theta(\boldsymbol{x}_0) - \phi)$. This implies that $\cos(\alpha - \phi) = \cos(\Theta(\boldsymbol{x}_0) - \phi)$ which implies $\alpha = 2\,k\,\pi + \Theta(\boldsymbol{x}_0)$ or $\alpha = 2\,k\,\pi + 2\,\phi - \Theta(\boldsymbol{x}_0)$, $k \in \mathbb{Z}$. To conclude $\boldsymbol{z}_0 = \boldsymbol{x}_0$, assume that $\boldsymbol{x}_0$ and $\boldsymbol{z}_0 = \|\boldsymbol{x}_0\|\,\breve{\boldsymbol{u}}(2\,\phi - \Theta(\boldsymbol{x}_0))$ are indistinguishable. Then, $h(\Phi_t^{\boldsymbol{u}}(\boldsymbol{x}_0) = h(\Phi_t^{\boldsymbol{u}}(\boldsymbol{z}_0))$ for every $t \in J$ implies that $\boldsymbol{c}^{\mathrm{T}}[v\,\omega^{-1}\left(\breve{\boldsymbol{u}}^{\perp}(\phi) - \breve{\boldsymbol{u}}^{\perp}(\omega\, t + \phi)\right) + \boldsymbol{v}_{\mathbf{c}_0}\, t] = 0$ for all $t \in [0, t_f)$ where $\boldsymbol{c} = \breve{\boldsymbol{u}}(\Theta(\boldsymbol{x}_0)) - \breve{\boldsymbol{u}}(2\,\phi - \Theta(\boldsymbol{x}_0))$. This implies that all the time derivatives are also zero. By evaluating the third and fourth time derivative at time $t = 0$, we have $[\breve{\boldsymbol{u}}(\Theta(\boldsymbol{x}_0)) - \breve{\boldsymbol{u}}(2\,\phi - \Theta(\boldsymbol{x}_0))]^{\mathrm{T}}\breve{\boldsymbol{u}}^{\perp}(\phi) = 0$ and $[\breve{\boldsymbol{u}}(\Theta(\boldsymbol{x}_0)) - \breve{\boldsymbol{u}}(2\,\phi - \Theta(\boldsymbol{x}_0))]^{\mathrm{T}}\breve{\boldsymbol{u}}(\phi) = 0$. From this we conclude that $\Theta(\boldsymbol{x}_0) = \phi$ and hence $\boldsymbol{z}_0 = \boldsymbol{x}_0$. This implies that the system is observable. This completes the proof.

Remark 5.14. Note that for a given initial condition $\boldsymbol{x}_0 \in \mathcal{M}$, every input $\chi \in \mathcal{U}_{\mathrm{con}}$ distinguish every other point $\boldsymbol{z} \in \mathcal{M}$ from $\boldsymbol{x}_0$.

The following corollary follows from propositions 5.10 and 5.13.

Corollary 5.15. Consider $\boldsymbol{x}_0 \in \mathcal{M}$. Then (5.5)-(5.6) is observable with respect to $\mathcal{U}_{\mathrm{con}}$ and $\mathcal{U}_{\mathrm{cir}}$.

5.3 CONSTANT, UNKNOWN OCEAN CURRENTS

Consider the single beacon positioning problem with constant, unknown ocean current

$$\begin{bmatrix}\dot{\boldsymbol{x}}(t)\\ \dot{\boldsymbol{v}}_\mathbf{c}(t)\end{bmatrix} = \begin{bmatrix}\boldsymbol{v}_\mathbf{c}(t)\\ \mathbf{0}\end{bmatrix} + \begin{bmatrix}v(t)\,\breve{\boldsymbol{u}}(\chi(t))\\ \mathbf{0}\end{bmatrix} \tag{5.7}$$

$$y(t) = \|\boldsymbol{x}(t)\|, \tag{5.8}$$

where the state $(\boldsymbol{x}, \boldsymbol{v}_\mathbf{c}) \in \mathcal{M} := \mathbb{R}^2 \setminus \{0\} \times \mathbb{R}^2$ and $\chi \in \mathcal{U}$ is the known relative course input, and $v \in \mathcal{V}$ is the known speed of the vehicle. Let $\mathcal{I}_3(\boldsymbol{x}_0, \boldsymbol{v}_{\mathbf{c}_0})$ denote the set of indistinguishable states from a given state $(\boldsymbol{x}_0, \boldsymbol{v}_{\mathbf{c}_0}) \in \mathcal{M}$ associated with system (5.7)-(5.8).

Assumption 5.16. We assume that v is a positive constant.

Constant relative course The solution of (5.7) for the inputs $\phi \in \mathcal{U}_{\mathrm{con}}$ and $v > 0$ corresponding to the initial state $(\boldsymbol{x}_0, \boldsymbol{v}_{\mathbf{c}_0}) \in \mathcal{M}$ is given by

$$\Phi_t^{\boldsymbol{u}}(\boldsymbol{x}_0, \boldsymbol{v}_{\mathbf{c}_0}) = \begin{bmatrix}\boldsymbol{x}_0\\ \boldsymbol{v}_{\mathbf{c}_0}\end{bmatrix} + \begin{bmatrix}\boldsymbol{v}_{\mathbf{t}_0}(\boldsymbol{u})\\ \mathbf{0}\end{bmatrix}\, t,$$

and the output is given by $h(\Phi_t^{\boldsymbol{u}}(\boldsymbol{x}_0, \boldsymbol{v}_{\mathbf{c}_0})) = \|\boldsymbol{x}_0 + \boldsymbol{v}_{\mathbf{t}_0}(\phi)\, t\|$ where $\boldsymbol{v}_{\mathbf{t}_0}(\boldsymbol{u}) \overset{\mathrm{def}}{=} v\,\breve{\boldsymbol{u}}(\phi) + \boldsymbol{v}_{\mathbf{c}_0}$. We have the following characterization for the unobservable subspace from a given initial position and current.

Proposition 5.17. Consider the system (5.7)-(5.8) and let $\mathcal{U}_{\mathrm{con}}$ be the class of admissible inputs. Consider $\mathbf{x}_0 := (\boldsymbol{x}_0, \boldsymbol{v}_{\mathbf{c}_0}) \in \mathcal{M}$ and define $\delta(\phi) \overset{\mathrm{def}}{=} \Theta(\boldsymbol{x}_0) - \Theta(\boldsymbol{v}_{\mathbf{t}_0}(\phi))$ and $\boldsymbol{v}_{\mathbf{t}_0}(\boldsymbol{u}) = v\,\breve{\boldsymbol{u}}(\phi) + \boldsymbol{v}_{\mathbf{c}_0}$ for every $\phi \in \mathcal{U}_{\mathrm{con}}$. Then, for every $\phi \in \mathcal{U}_{\mathrm{con}}$,

$$\mathcal{I}_3(\mathbf{x}_0) = \Big\{(\|\boldsymbol{x}_0\|\breve{\boldsymbol{u}}(\alpha + \delta(\phi)), \|\boldsymbol{v}_{\mathbf{t}_0}(\boldsymbol{u})\|\,\breve{\boldsymbol{u}}(\alpha) - v\,\breve{\boldsymbol{u}}(\phi)) : \alpha \in [0, 2\pi)\Big\}$$
$$\bigcup \{(\|\boldsymbol{x}_0\|\breve{\boldsymbol{u}}(\alpha - \delta(\phi)), \|\boldsymbol{v}_{\mathbf{t}_0}(\boldsymbol{u})\|\,\breve{\boldsymbol{u}}(\alpha) - v\,\breve{\boldsymbol{u}}(\phi)) \colon \alpha \in [0, 2\pi)\Big\}.$$

Proof. Consider $\mathbf{x}_0 := (\boldsymbol{x}_0, \boldsymbol{v}_{\mathbf{c}_0}) \in \mathcal{M}$ and let $\mathbf{z}_0 := (\boldsymbol{z}_0, \boldsymbol{w}_{\mathbf{c}_0}) \in \mathcal{I}_3(\mathbf{x}_0)$. Then $h(\Phi_t^{\boldsymbol{u}}(\mathbf{z}_0)) = h(\Phi_t^{\boldsymbol{u}}(\mathbf{x}_0))$ for all $t \in J$ and $\chi \in \mathcal{U}_{\mathrm{con}}$, $v > 0$ which implies that

$$\|\boldsymbol{z}_0\| = \|\boldsymbol{x}_0\|, \tag{5.9}$$

$$\|\boldsymbol{w}_{\mathbf{t}_0}(\phi)\| = \|\boldsymbol{v}_{\mathbf{t}_0}(\phi)\|, \tag{5.10}$$

$$\boldsymbol{z}_0^{\mathrm{T}}\boldsymbol{w}_{\mathbf{t}_0}(\phi) = \boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{v}_{\mathbf{t}_0}(\phi), \tag{5.11}$$

where $\boldsymbol{w}_{\mathbf{t}_0}(\boldsymbol{u}) = v\,\breve{\boldsymbol{u}}(\phi) + \boldsymbol{w}_{\mathbf{c}_0}$. Note that we have four unknown variables $(\boldsymbol{z}_0, \boldsymbol{w}_{\mathbf{c}_0})$ and three equations. Hence the solution space is one-dimensional manifold and requires one parameter to parametrize the solution space. Now $\|\boldsymbol{z}_0\| = \|\boldsymbol{x}_0\|$ and $\|\boldsymbol{w}_{\mathbf{t}_0}(\boldsymbol{u})\| = \|\boldsymbol{v}_{\mathbf{t}_0}(\boldsymbol{u})\|$ are respectively equivalent to

$$G_z := \{\|\boldsymbol{x}_0\|\,\breve{\boldsymbol{u}}(\alpha_z) \in \mathbb{R}^2 :\ \alpha_z \in [0, 2\,\pi)\},$$
$$G_\mathbf{t} := \{\|\boldsymbol{v}_{\mathbf{t}_0}(\boldsymbol{u})\|\,\breve{\boldsymbol{u}}(\alpha_\mathrm{c}) \in \mathbb{R}^2 :\ \alpha_\mathrm{c} \in [0, 2\,\pi)\}.$$

Since $\boldsymbol{z}_0 := \|\boldsymbol{x}_0\|\,\breve{\boldsymbol{u}}(\alpha_z) \in G_z$ and $\boldsymbol{w}_{\mathbf{t}_0} := \|\boldsymbol{v}_{\mathbf{t}_0}(\boldsymbol{u})\|\,\breve{\boldsymbol{u}}(\alpha_\mathbf{c}) \in G_\mathbf{t}$, the third condition $\boldsymbol{z}_0^{\mathrm{T}}\boldsymbol{w}_{\mathbf{t}_0} = \boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{v}_{\mathbf{t}_0}$ implies that $\cos(\alpha_z - \alpha_\mathbf{c}) = \cos\delta_0$, where $\delta_0 = \Theta(\boldsymbol{x}_0) - \Theta(\boldsymbol{v}_{\mathbf{t}_0}(\boldsymbol{u}))$, that is, either $\alpha_z - \alpha_{\mathbf{c}} = 2\,k\,\pi + \delta_0$ or $\alpha_z - \alpha_{\mathbf{c}} = 2\,k\,\pi - \delta_0$. In other words, either $\alpha_z = 2\,k\,\pi + (\alpha_{\mathbf{c}} + \delta_0)$ or $\alpha_z = 2\,k\,\pi + (\alpha_{\mathbf{c}} - \delta_0)$.

Note that G_z and $G_{\mathbf{t}}$ are parametrized by two variables α_z and $\alpha_{\mathbf{c}}$, respectively. Further $p := \|\boldsymbol{v}_{\mathbf{t}_0}\|\,\boldsymbol{u}(\alpha_{\mathbf{c}}) = v\,\breve{\boldsymbol{u}}(\phi) + \boldsymbol{w}_{\mathbf{c}_0} \in G_{\mathbf{t}}$, from which $\boldsymbol{w}_{\mathbf{c}_0}$ can be parametrized by $\alpha_{\mathbf{c}}$ as

$$G_{\mathbf{c}} = \{\|\boldsymbol{v}_{\mathbf{t}_0}\|\,\breve{\boldsymbol{u}}(\alpha_{\mathbf{c}}) - v\,\breve{\boldsymbol{u}}(\phi) \in \mathbb{R}^2 :\ \alpha_{\mathbf{c}} \in [0, 2\pi)\}.$$

Thus we have shown that $\mathbf{z}_0 \in \mathcal{I}_3(\mathbf{x}_0)$ implies that $\mathbf{z}_0 \in \mathcal{J}$, and consequently $\mathcal{I}_3(\mathbf{x}_0) \subseteq \mathcal{J}$, where

$$\mathcal{J} := \{(\|\boldsymbol{x}_0\|\breve{\boldsymbol{u}}(\alpha_{\mathbf{c}} + \delta_0), \|\boldsymbol{v}_{\mathbf{t}_0}(\boldsymbol{u})\|\breve{\boldsymbol{u}}(\alpha_{\mathbf{c}}) - v\breve{\boldsymbol{u}}(\phi)) : \alpha_{\mathbf{c}} \in [0, 2\pi)\}$$
$$\bigcup \{(\|\boldsymbol{x}_0\|\,\breve{\boldsymbol{u}}(\alpha_{\mathbf{c}} - \delta_0), \|\boldsymbol{v}_{\mathbf{t}_0}(\boldsymbol{u})\|\,\breve{\boldsymbol{u}}(\alpha_{\mathbf{c}}) - v\,\breve{\boldsymbol{u}}(\phi)) :\ \alpha_{\mathbf{c}} \in [0, 2\pi)\}.$$

In order to show the reverse inclusion, consider $\mathbf{z}_0 \in \mathcal{J}$. Then either $\mathbf{z}_0 = (\|\boldsymbol{x}_0\|\breve{\boldsymbol{u}}(\alpha_{\mathbf{c}} + \delta_0), \|\boldsymbol{v}_{\mathbf{t}_0}(\boldsymbol{u})\|\breve{\boldsymbol{u}}(\alpha_{\mathbf{c}}) - v\breve{\boldsymbol{u}}(\phi))$ or $\mathbf{z}_0 = (\|\boldsymbol{x}_0\|\,\breve{\boldsymbol{u}}(\alpha_{\mathbf{c}} - \delta_0), \|\boldsymbol{v}_{\mathbf{t}_0}(\boldsymbol{u})\|\,\breve{\boldsymbol{u}}(\alpha_{\mathbf{c}}) - v\,\breve{\boldsymbol{u}}(\phi))$, where $\alpha_{\mathbf{c}} \in [0, 2\pi)$, $v > 0$ and $\phi \in \mathcal{U}_{\text{con}}$. Now it can be easily verified that for both the cases

$$h(\Phi_t^{\boldsymbol{u}}(\mathbf{z}_0)) = \|\boldsymbol{x}_0 + \boldsymbol{v}_{\mathbf{t}_0}(\boldsymbol{u})\,t\| = h(\Phi_t^{\boldsymbol{u}}(\mathbf{x}_0))$$

for every $t \in J$, $v > 0$, and $\chi \in \mathcal{U}_{\text{con}}$. Consequently, $\mathcal{J} \subseteq \mathcal{I}_3(\mathbf{x}_0)$. Hence the equality follows.

Constant (nonzero) relative course rate

Proposition 5.18. Consider the system (5.7)-(5.8) and let $\mathcal{U}_{\text{cir}}$ be the class of admissible inputs. Then, $\mathcal{I}_3(\mathbf{x}_0) = \{\mathbf{x}_0\}$ for every $\mathbf{x}_0 \in \mathcal{M}$.

Proof. Consider $\mathbf{x}_0 := (\boldsymbol{x}_0, \boldsymbol{v}_{\mathbf{c}_0}) \in \mathcal{M}$ and let $\mathbf{z}_0 := (\boldsymbol{z}, \boldsymbol{w}_{\mathbf{c}}) \in \mathcal{I}(\mathbf{x}_0)$. Then $h(\Phi_t^{\boldsymbol{u}}(\mathbf{x}_0)) = h(\Phi_t^{\boldsymbol{u}}(\mathbf{z}_0))$ for all $t \in J$, $v > 0$ and $\chi \in \mathcal{U}_{\text{cir}}$. In particular at $t = 0$, we have $\|\boldsymbol{z}\| = \|\boldsymbol{x}_0\|$ or, equivalently, $\boldsymbol{z} = \|\boldsymbol{x}_0\|\,\breve{\boldsymbol{u}}(\beta)$, $\beta \in [0, 2\pi)$.

Choose $v_0, \omega_0 > 0$ and $\phi_0 \in [0, 2\pi)$ such that $\chi(t) = \omega_0\,t + \phi_0$. Define $t \mapsto g(t)$ given by $g(t) := h(\Phi_t^{\boldsymbol{u}}(\mathbf{x}_0)) - h(\Phi_t^{\boldsymbol{u}}(\mathbf{z}_0))$, $t \in J$. Note that g is a function of t and we have the condition $g(t) \equiv 0$. This implies all the higher order derivatives of g at $t = 0$ are zero. The third, fourth, fifth, and sixth time derivatives of g at $t = 0$ are given by

$$\boldsymbol{w}_{\mathbf{c}}^{\mathrm{T}}\breve{\boldsymbol{u}}^{\perp}(\phi_0) = \boldsymbol{v}_{\mathbf{c}_0}^{\mathrm{T}}\breve{\boldsymbol{u}}^{\perp}(\phi_0) - \frac{\omega_0}{3}\,(\boldsymbol{x}_0 - \boldsymbol{z}_0)^{\mathrm{T}}\breve{\boldsymbol{u}}(\phi_0), \qquad (5.12)$$

$$\boldsymbol{w}_{\mathbf{c}}^{\mathrm{T}}\breve{\boldsymbol{u}}(\phi_0) = v_{\mathbf{c}_0}^{\mathrm{T}}\breve{\boldsymbol{u}}(\phi_0) + \frac{\omega_0}{4}\,(\boldsymbol{x}_0 - \boldsymbol{z}_0)^{\mathrm{T}}\breve{\boldsymbol{u}}^{\perp}(\phi_0), \qquad (5.13)$$

$$\boldsymbol{w}_{\mathbf{c}}^{\mathrm{T}}\breve{\boldsymbol{u}}^{\perp}(\phi_0) = \boldsymbol{v}_{\mathbf{c}_0}^{\mathrm{T}}\breve{\boldsymbol{u}}^{\perp}(\phi_0) - \frac{\omega_0}{5}\,(\boldsymbol{x}_0 - \boldsymbol{z}_0)^{\mathrm{T}}\breve{\boldsymbol{u}}(\phi_0), \qquad (5.14)$$

$$\boldsymbol{w}_{\mathbf{c}}^{\mathrm{T}}\breve{\boldsymbol{u}}(\phi_0) = \boldsymbol{v}_{\mathbf{c}_0}^{\mathrm{T}}\breve{\boldsymbol{u}}(\phi_0) + \frac{\omega_0}{6}\,(\boldsymbol{x}_0 - \boldsymbol{z}_0)^{\mathrm{T}}\breve{\boldsymbol{u}}^{\perp}(\phi_0). \qquad (5.15)$$

Since $\omega_0 > 0$, equations (5.12) and (5.14) imply that $(x_0 - z_0)^{\mathrm{T}}\breve{\boldsymbol{u}}(\phi_0) = 0$, while (5.13) and (5.15) imply that $(x_0 - z_0)^{\mathrm{T}}\breve{\boldsymbol{u}}^{\perp}(\phi_0) = 0$. Since $\{\breve{\boldsymbol{u}}(\phi_0), \breve{\boldsymbol{u}}^{\perp}(\phi_0)\}$ is linearly independent, $(x_0 - z_0)^{\mathrm{T}}\breve{\boldsymbol{u}}(\phi_0) = 0$ and $(x_0 - z_0)^{\mathrm{T}}\breve{\boldsymbol{u}}^{\perp}(\phi_0) = 0$ imply that $z_0 = x_0$. Consequently, (5.12) and (5.13) imply that $w_{\mathbf{c}} = v_{\mathbf{c}_0}$. Thus we have shown that $\mathbf{z}_0 \in \mathcal{I}(\mathbf{x}_0)$ implies $\mathbf{z}_0 = \mathbf{x}_0$ and hence $\mathcal{I}(\mathbf{x}_0) \subseteq \{\mathbf{x}_0\}$. Since v_0, ω_0, and ϕ_0 were chosen to be arbitrary, we conclude that $\mathcal{I}(\mathbf{x}_0) \subseteq \{\mathbf{x}_0\}$. Hence the result follows.

The following corollary follows from propositions 5.17 and 5.18.

Corollary 5.19. Consider $\boldsymbol{x}_0 \in \mathcal{M}$. Then (5.7)-(5.8) is not locally weakly observable with respect to $\mathcal{U}_{\text{con}}$ and is observable with respect to $\mathcal{U}_{\text{cir}}$.

We summarize the results in Table 1.

6. CONCLUSIONS

In general, it is difficult to obtain a necessary and sufficient condition for the observability of the single-beacon navigation system due to the fact that the observability of the system depends on the class of admissible relative course inputs. In this paper we studied the observability properties of 2D single-beacon navigation using range measurements for two class of inputs, namely, constant relative course and constant relative course rate. We have shown that for relative constant relative course the system with known constant ocean currents is globally observable in the sense of Herman and Krener. On the other hand, the system fails to be locally observable with unknown ocean currents for relative constant relative course inputs, where as global observability can be achieved with a relative constant nonzero relative course rate. Thus we have elaborated that with a simple class of relative course inputs it is possible to achieve global observability. Importantly, by the concatenation of constant relative course and constant relative course rate inputs (concatenation of line and circular arcs), it is possible to achieve an observable system. Future work involves characterizing set of admissible inputs that make the system globally observable and extending the results to 3D.

System	Straight line motion	Circular motion
No ocean current	O	O
Known ocean current	O	O
Unknown ocean current	Neither LWO nor O	O

Table 1. Observability analysis for constant relative course and constant (nonzero) relative course rate

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