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Further results on the observability in magneto-inertial navigation

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Abstract— **This paper shows that one can relax an important assumption usually formulated to employ the magneto-inertial navigation (MINAV) technique. This technique, which allows to reconstruct the velocity of a rigid body moving in a magnetically disturbed area, usually assumes that the (unknown) Jacobian of the magnetic field is everywhere non-singular. As it is here demonstrated, this assumption can be significantly alleviated thanks to further investigations on the observability of the dynamics at stake. Corresponding relaxed assumptions require rotations to sufficiently come into play during the motion one wishes to estimate the velocity of. This result opens new perspectives on the range of applicability of the MINAV technique.**

I. Introduction

Recently, a new method to estimate the motion of a rigid body moving in a magnetically disturbed environment has been proposed under the name "magneto-inertial navigation" (MINAV) [1], [2], [3]. As its name calls to mind, it relies on the combined usage of magnetic and inertial sensors. Briefly, this methods relates the variations of the sensed field to the translational and rotational velocities of the rigid body. Eventually, the velocities are integrated over time to estimate the trajectory of the body under consideration.

In short, this technique exploits the main equation governing the dynamics of the magnetic field, expressed in the frame of the sensors, and that depends on the linear and angular velocity, the magnetic field itself, and the Jacobian of the magnetic field, all expressed in body-fixed coordinates. All quantities but the linear velocity are readily available: the angular velocity is measured by a set of rate gyros or an Attitude and Heading Reference System (AHRS); the magnetic field is given by a magnetometer; and the Jacobian matrix is given by a custom-built magnetic gradiometer consisting of a set of spatially distributed magnetometers. The only unknown term is the linear velocity, which can thus be estimated based on the knowledge of the other terms and the governing system dynamics. In order to estimate the linear velocity, state observers are employed and several versions, relying on various modeling assumptions for the unknown linear velocity, have been proposed in [4] and [5]. Their analysis are usually performed by a point-wise analysis of observability [5] or a Lyapunov analysis [3], calling for careful application of the invariance principle in linear time-varying cases.

In all aforementioned works, a central assumption has been formulated, which amounts to the requirement that the Jacobian of the magnetic field should be non-singular over the whole region of navigation interest. Practically, this assumption is usually guaranteed, at least to a certain practical extent, as a very large number of magnetic disturbances are commonly present in the areas of interest. Successful experimentations have been reported and stress that MINAV is a practically embeddable technology, being capable of e.g. reconstructing the motion of a pedestrian walking inside a building (where GPS signal are not available) with an accuracy below 5% of error over tens of minutes of walking.

Yet, it is of great interest to alleviate the assumption that the Jacobian should be at all times non-singular. As it has been studied in less favorable cases [3], singularities can occur and it remains relatively unclear what their impact on observers convergence is. As it will be shown in this article, the fundamental assumption can be relaxed by imposing very mild conditions on the trajectory of the rigid body itself. This result, which is the main contribution of the article, is related to recent results on observability of linear time-varying (LTV) systems [6] for navigation systems. Interestingly, these results find here a relatively direct case of application, which, in turn, reveals some valuable perspective on the usage and generalizations of MINAV systems.

The paper is organized as follows. The problem under consideration is stated in Section II, where the fundamental equations of the MINAV technique are recalled and two cases of particular interest are introduced: *i)* constant velocity in the body-fixed frame of reference; and *ii)* constant velocity in the inertial frame. In addition, a reduced sensor suite setting is also described, yielding a total of 4 different models in the end. In Section III, the tools of observability analysis for LTV systems are introduced and used to establish two properties assessing the observability of the MINAV dynamics for all cases previously introduced. In Section IV a simple illustrative example is treated which gives a pictorial view of the relaxed assumption. Finally, some concluding remarks are presented in Section V.

Notations

The symbol $\mathbf{0}$ denotes a matrix (or vector) of zeros, and $\mathbf{I}$ the identity matrix. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, the cross product is represented by $\mathbf{x} \times \mathbf{y}$.

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II. Problem statement

A. System dynamics

In order to set the problem framework, let $\{I\}$ denote an inertial reference frame and consider a mobile platform which has attached to it the so-called body-fixed reference frame, denoted as $\{B\}$. The mobile platform is assumed to be equipped with a triad of orthogonally mounted rate gyros, which measure the angular velocity of the mobile platform with respect to the inertial frame, expressed in body-fixed coordinates, denoted as $\boldsymbol{\omega}(t) \in \mathbb{R}^3$. In addition, a set of magnetometers is also installed on-board, which allow to measure the magnetic field, also expressed in body-fixed coordinates, denoted as $\mathbf{h}(t) \in \mathbb{R}^3$.

In the absence of magnetic disturbances, the magnetic field can be assumed locally constant in inertial coordinates and as such the derivative of the body-fixed measurements is simply given by $\dot{\mathbf{h}}(t) = -\mathbf{S}\left(\boldsymbol{\omega}(t)\right)\mathbf{h}(t)$, where $\mathbf{S}\left(.\right)$ is the skew-symmetric matrix that encodes the cross product, i.e., $\mathbf{S}\left(\boldsymbol{x}\right)\boldsymbol{y} \triangleq \boldsymbol{x}\times\boldsymbol{y}$, with $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$. When there exist magnetic disturbances or artificially generated magnetic fields, the dynamic equation governing the evolution of the inertial magnetic field can be written, using the chain rule, as

$$^{I}\dot{\mathbf{h}}(t) = \mathbf{J}_i\left(t, \mathbf{p}(t)\right){}^{I}\mathbf{v}(t),$$

where $^{I}\mathbf{h}(t) \in \mathbb{R}^3$ is the inertial magnetic field, $^{I}\mathbf{v}(t) \in \mathbb{R}^3$ is the inertial linear velocity of the mobile platform, and $\mathbf{J}_i\left(t, \mathbf{p}(t)\right) \in \mathbb{R}^{3\times3}$ is the Jacobian of the inertial magnetic field, which depends on the inertial position of the mobile platform, denoted as $\mathbf{p}(t) \in \mathbb{R}^3$. The magnetic field, expressed in body-fixed coordinates, is simply given by

$$\mathbf{h}(t) = \mathbf{R}^T(t){}^{I}\mathbf{h}(t), \tag{1}$$

where $\mathbf{R}(t) \in SO(3)$ stands for the rotation matrix from $\{B\}$ to $\{I\}$. Taking the derivative of (1) gives

$$\dot{\mathbf{h}}(t) = -\mathbf{S}\left(\boldsymbol{\omega}(t)\right)\mathbf{h}(t) + \mathbf{J}\left(t\right)\mathbf{v}(t),$$

where $\mathbf{v}(t) \in \mathbb{R}^3$ is the linear velocity of the mobile platform, expressed in body-fixed coordinates, and $\mathbf{J}(t) \in \mathbb{R}^{3\times3}$ is the Jacobian of the magnetic field, expressed in body-fixed coordinates. It is related to the Jacobian of the inertial magnetic field as [7]

$$\mathbf{J}(t) = \mathbf{R}^T(t)\mathbf{J}_i\left(t, \mathbf{p}(t)\right)\mathbf{R}(t).$$

The problem considered here is the estimation of the linear velocity of the mobile platform. Without additional sensors, two nominal models can be assumed: *i)* constant velocity expressed in body-fixed coordinates; and *ii)* constant velocity in inertial coordinates. From the practical point of view, and considering a stochastic filtering formalism, it is possible to consider, e.g., that the velocity models are driven by zero-mean white Gaussian processes. This allows to capture slowly time-varying linear velocities. In the ensuing, a deterministic setting is first considered for observability analysis.

While the choice of the models for the linear velocity may seem harmless at first glance, this choice yields differences in terms of the observability of the system[1]. In this paper, even though both models are studied in careful detail, the model of constant linear velocity in body-fixed coordinates is developed first. This choice is justified from practical considerations, and the application to pedestrian indoor navigation [3]: indeed, while a mobile platform may change its attitude a lot (e.g. an in-door mobile platform or a car can easily change its orientation), its velocity may remains relatively constant for rather long periods of time (several seconds).

Considering a constant body-fixed velocity model, the system dynamics are given by

$$\left\{\begin{array}{l} \dot{\mathbf{h}}(t) = -\mathbf{S}\left(\boldsymbol{\omega}(t)\right)\mathbf{h}(t) + \mathbf{J}\left(t\right)\mathbf{v}(t) \\ \dot{\mathbf{v}}(t) = \mathbf{0} \end{array}\right. . \tag{2}$$

On the other hand, considering a constant inertial velocity model, the system dynamics are given by

$$\left\{\begin{array}{l} \dot{\mathbf{h}}(t) = -\mathbf{S}\left(\boldsymbol{\omega}(t)\right)\mathbf{h}(t) + \mathbf{J}\left(t\right)\mathbf{v}(t) \\ \dot{\mathbf{v}}(t) = -\mathbf{S}\left(\boldsymbol{\omega}(t)\right)\mathbf{v}(t) \end{array}\right. . \tag{3}$$

B. Possible reduction of the sensor suite

In order to employ the models (2) or (3) for linear velocity estimation, the full Jacobian $\mathbf{J}$ is required, which means that a set of four triaxial magnetometers is needed for implementation. However, it is possible to estimate the linear velocity resorting only to the governing dynamics of one component of the magnetic field. In turn, this means that only one row of the Jacobian is needed, and hence less magnetometers are required.

One such configuration will be studied. In this paper, and without loss of generality, it is assumed for the reduced sensor suite setting that only the governing equation of the magnetic field along the x-axis is employed. In this case, only one triaxial and three single-axis magnetometers are required, which allow to measure the magnetic field along the x-axis and to estimate the first row of $\mathbf{J}(t)$ with a custom-built magnetic gradiometer, consisting of the three single-axis magnetometers and the x component of the triaxial magnetometer.

The system dynamics for the reduced sensor suite setting, considering a constant body-fixed velocity model, are given by

$$\left\{\begin{array}{l} \dot{h}_x(t) = \mathbf{J}_x(t)\mathbf{v}(t) - \omega_z(t)h_y(t) + \omega_y(t)h_z(t) \\ \dot{\mathbf{v}}(t) = \mathbf{0} \end{array}\right. , \tag{4}$$

where

$$\mathbf{h}(t) = \left[\begin{array}{c} h_x(t) \\ h_y(t) \\ h_z(t) \end{array}\right] \in \mathbb{R}^3,\ h_x(t), h_y(t), h_z(t) \in \mathbb{R},$$

$$\boldsymbol{\omega}(t) = \left[\begin{array}{c} \omega_x(t) \\ \omega_y(t) \\ \omega_z(t) \end{array}\right] \in \mathbb{R}^3,\ \omega_x(t), \omega_y(t), \omega_z(t) \in \mathbb{R},$$

and

$$\mathbf{J}(t) = \left[\begin{array}{c} \mathbf{J}_x(t) \\ \mathbf{J}_y(t) \\ \mathbf{J}_z(t) \end{array}\right],$$

[1] the models also correspond to distinct practical situations.

with $\mathbf{J}_x(t), \mathbf{J}_y(t), \mathbf{J}_z(t) \in \mathbb{R}^{1\times 3}$. On the other hand, if a constant inertial velocity model is considered with the reduced sensor suite, the system dynamics read

$$\left\{ \begin{array}{l} \dot{h}_x(t) = \mathbf{J}_x(t)\mathbf{v}(t) - \omega_z(t)h_y(t) + \omega_y(t)h_z(t) \\ \dot{\mathbf{v}}(t) = -\mathbf{S}\left(\boldsymbol{\omega}(t)\right)\mathbf{v}(t) \end{array} \right. . \quad (5)$$

All quantities are assumed bounded, which is a mild assumption that is always verified in practice.

III. Observability Analysis

This section details the observability analysis, at large, of the system dynamics introduced in Section II. The following real-analysis result [Proposition 4.2, [6]] is useful in the sequel.

Lemma 1: Let $\mathbf{f}(t) : [t_0, t_f] \subset \mathbb{R} \to \mathbb{R}^n$ be a continuous and i-times continuously differentiable function on $\mathcal{I} := [t_0, t_f]$, and such that

$$\mathbf{f}(t_0) = \dot{\mathbf{f}}(t_0) = \ldots = \mathbf{f}^{(i-1)}(t_0) = \mathbf{0}.$$

Further assume that $\left\|\mathbf{f}^{(i+1)}(.)\right\|$ is bounded on $\mathcal{I}$. If

$$\underset{\substack{\alpha>0 \\ t_1\in\mathcal{I}}}{\exists} : \left\|\mathbf{f}^{(i)}(t_1)\right\| \geq \alpha,$$

then

$$\underset{\substack{0<\delta\leq t_f-t_0 \\ \beta>0}}{\exists} : \left\|\mathbf{f}(t_0+\delta)\right\| \geq \beta.$$

A. Constant body-fixed velocity model

In compact form, it is possible to rewrite (2) as

$$\left\{ \begin{array}{l} \dot{\mathbf{x}}_1(t) = \mathbf{A}_1(t)\mathbf{x}_1(t) \\ \mathbf{y}_1(t) = \mathbf{C}_1\mathbf{x}_1(t) \end{array} \right. , \quad (6)$$

where $\mathbf{x}_1(t) = \begin{bmatrix}\mathbf{h}^T(t) & \mathbf{v}^T(t)\end{bmatrix}^T \in \mathbb{R}^6$ is the system state,

$$\mathbf{A}_1(t) = \begin{bmatrix} -\mathbf{S}\left(\boldsymbol{\omega}(t)\right) & \mathbf{J}(t) \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{6\times 6},$$

and $\mathbf{C}_1 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{3\times 6}$, which is a linear time-varying system. The following proposition establishes a necessary and sufficient condition on the observability (in the sense of non-singularity of the observability Grammian) of (6).

Proposition 1: The LTV system (6) is observable on $\mathcal{I}$ if and only if, for all unit vectors $\mathbf{d} \in \mathbb{R}^3$, it is possible to choose $t_i \in \mathcal{I}$, $\mathcal{I} := [t_0, t_f]$, such that $\mathbf{J}(t_i)\mathbf{d} \neq 0$.

Proof: First, as observability is preserved under output feedback, observability of the pair $(\mathbf{A}_1(t), \mathbf{C}_1)$ is equivalent to observability of the pair $(\boldsymbol{\mathcal{A}}_1(t), \mathbf{C}_1)$, where

$$\boldsymbol{\mathcal{A}}_1(t) = \begin{bmatrix} \mathbf{0} & \mathbf{J}(t) \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{6\times 6}.$$

The transition matrix associated with $\boldsymbol{\mathcal{A}}_1(t)$ is simply given by

$$\boldsymbol{\phi}_1(t, t_0) = \begin{bmatrix} \mathbf{I} & \int_{t_0}^{t} \mathbf{J}(\sigma)\, d\sigma \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \in \mathbb{R}^{6\times 6}.$$

Let $\mathbf{c} = \begin{bmatrix}\mathbf{c}_1^T & \mathbf{c}_2^T\end{bmatrix}^T \in \mathbb{R}^6$ be any unit vector and $\boldsymbol{\mathcal{W}}_1(t_0, t_f)$ the observability Grammian associated with the pair $(\boldsymbol{\mathcal{A}}_1(t), \mathbf{C}_1)$. Then,

$$\begin{aligned} \mathbf{c}^T\boldsymbol{\mathcal{W}}_1(t_0, t_f)\,\mathbf{c} &= \int_{t_0}^{t_f} \left\|\mathbf{C}_1(t)\boldsymbol{\phi}_1(t, t_0)\,\mathbf{c}\right\|^2 dt \\ &= \int_{t_0}^{t_f} \left\|\mathbf{f}_1(t, t_0)\right\|^2 dt, \end{aligned}$$

where

$$\mathbf{f}_1(t, t_0) = \mathbf{c}_1 + \left(\int_{t_0}^{t} \mathbf{J}(\sigma)\, d\sigma\right)\mathbf{c}_2.$$

In order to show necessity, suppose that there exists a unit vector $\mathbf{d} \in \mathbb{R}^3$ such that, for all $t \in \mathcal{I}$, $\mathbf{J}(t)\mathbf{d} = \mathbf{0}$. Let $\mathbf{c} = \begin{bmatrix}\mathbf{0} & \mathbf{d}^T\end{bmatrix}^T$. Then, it is trivial to conclude that $\mathbf{f}_1(t, t_0) = \mathbf{0}$ for all $t \in \mathcal{I}$ and hence $\mathbf{c}^T\boldsymbol{\mathcal{W}}_1(t_0, t_f)\,\mathbf{c} = 0$, which means that the observability Grammian $\boldsymbol{\mathcal{W}}_1(t_0, t_f)$ is not invertible which contradicts the observability assumption. Thus, if the LTV system is observable on $\mathcal{I}$, it must be true that, for all unit vectors $\mathbf{d} \in \mathbb{R}^3$, it is possible to choose $t_i \in \mathcal{I}$ such that $\mathbf{J}(t_i)\mathbf{d} \neq 0$.

In order to show sufficiency, we need to consider every possible unit vector $\mathbf{c} = \begin{bmatrix}\mathbf{c}_1^T & \mathbf{c}_2^T\end{bmatrix}^T$. First, assume that $\mathbf{c}_1 \neq \mathbf{0}$. In that case, $\left\|\mathbf{f}_1(t_0, t_0)\right\| = \left\|\mathbf{c}_1\right\| > 0$ and hence, using Lemma 1, there exists $t_1 \in \,]t_0, t_f]$ such that $\mathbf{c}^T\boldsymbol{\mathcal{W}}_1(t_0, t_1)\,\mathbf{c} > 0$. Suppose now that $\mathbf{c}_1 = \mathbf{0}$. As $\mathbf{c}$ is a unit vector, it follows that in that case $\mathbf{c}_2$ is also a unit vector and $\mathbf{f}_1(t_0, t_0) = \mathbf{0}$. But now

$$\frac{\partial}{\partial t}\mathbf{f}_1(t, t_0) = \mathbf{J}(t)\mathbf{c}_2.$$

Under the conditions of the proposition, for all unit vectors $\mathbf{c}_2$, it is possible to choose a time instant $t_i \in \mathcal{I}$ such that

$$\left\| \left.\frac{\partial}{\partial t}\mathbf{f}_1(t, t_0)\right|_{t=t_i} \right\| > 0.$$

Then, using Lemma 1 again, it follows that, for all unit vectors $\mathbf{c}$ with $\mathbf{c}_1 = \mathbf{0}$, it is possible to choose $t_j \in \,]t_0, t_f]$ such that $\mathbf{c}^T\boldsymbol{\mathcal{W}}_1(t_0, t_j)\,\mathbf{c} > 0$. Thus it is shown that, for all unit vectors $\mathbf{c}$ there exists $t_k \in \,]t_0, t_j] \subset]t_0, t_f]$ such that $\mathbf{c}^T\boldsymbol{\mathcal{W}}_1(t_0, t_k)\,\mathbf{c} > 0$, which implies that, for all unit vectors $\mathbf{c}$, $\mathbf{c}^T\boldsymbol{\mathcal{W}}_1(t_0, t_f)\,\mathbf{c} > 0$. This concludes the proof as it is shown that, under the conditions of the proposition, the observability Grammian $\boldsymbol{\mathcal{W}}_1(t_0, t_f)$ is invertible. ■

To design state observers with globally asymptotically stable error dynamics, stronger conditions are required. Namely, Uniform Complete Observability (UCO) is sufficient to guarantee exponential convergence of the Kalman filter as can be established from the classic results of [8], [9], [10]. More discussions can be found in [11]. Using similar arguments of proofs, one can establish that the LTV system (6) is UCO if and only if there exist positive constants $\alpha > 0$ and $\delta > 0$ such that, for all $t \geq t_0$ and all unit vectors $\mathbf{d} \in \mathbb{R}^3$, it is possible to choose $t_i \in [t, t+\delta]$ such that $\left\|\mathbf{J}(t_i)\,\mathbf{d}\right\| \geq \alpha$. One needs to consider uniform bounds and use the fact uniform complete observability is preserved under bounded continuous output feedback, see [12, Lemma 4.8.1].

B. Constant inertial velocity model

In compact form, it is possible to rewrite (3) as

$$\left\{ \begin{array}{l} \dot{\mathbf{x}}_2(t) = \mathbf{A}_2(t)\mathbf{x}_2(t) \\ \mathbf{y}_2(t) = \mathbf{C}_2\mathbf{x}_2(t) \end{array} \right. , \quad (7)$$

where $\mathbf{x}_2(t) = \begin{bmatrix}\mathbf{h}^T(t) & \mathbf{v}^T(t)\end{bmatrix}^T \in \mathbb{R}^6$ is the system state,

$$\mathbf{A}_2(t) = \begin{bmatrix} -\mathbf{S}\left(\boldsymbol{\omega}(t)\right) & \mathbf{J}(t) \\ \mathbf{0} & -\mathbf{S}\left(\boldsymbol{\omega}(t)\right) \end{bmatrix} \in \mathbb{R}^{6\times 6},$$

and $\mathbf{C}_2 = \left[\begin{array}{cc} \mathbf{I} & \mathbf{0} \end{array} \right] \in \mathbb{R}^{3\times 6}$, which is a linear time-varying system. The following proposition establishes a necessary and sufficient condition on the observability of (7).

Proposition 2: The LTV system (7) is observable on $\mathcal{I}$ if and only if, for all unit vectors $\mathbf{d} \in \mathbb{R}^3$, it is possible to choose $t_i \in \mathcal{I}$, $\mathcal{I} := [t_0, t_f]$, such that $\mathbf{J}(t_i)\mathbf{R}^T(t_i)\mathbf{R}(t_0)\mathbf{d} \neq 0$.

Proof: Again, one can use the fact that observability is preserved under output feedback. Instead of the pair $(\mathbf{A}_2(t), \mathbf{C}_2)$, one can equivalently study the pair $(\boldsymbol{\mathcal{A}}_2(t), \mathbf{C}_2)$, where

$$\boldsymbol{\mathcal{A}}_2(t) = \left[\begin{array}{cc} \mathbf{0} & \mathbf{J}(t) \\ \mathbf{0} & -\mathbf{S}(\boldsymbol{\omega}(t)) \end{array} \right] \in \mathbb{R}^{6\times 6}.$$

The transition matrix associated with $\boldsymbol{\mathcal{A}}_2(t)$ is simply given by

$$\boldsymbol{\phi}_2(t, t_0) = \left[\begin{array}{cc} \mathbf{I} & \int_{t_0}^{t} \mathbf{J}(\sigma)\mathbf{R}^T(\sigma)\mathbf{R}(t_0)\, d\sigma \\ \mathbf{0} & \mathbf{R}^T(t)\mathbf{R}(t_0) \end{array} \right] \in \mathbb{R}^{6\times 6}.$$

Let $\mathbf{c} = \left[\begin{array}{cc} \mathbf{c}_1^T & \mathbf{c}_2^T \end{array} \right]^T \in \mathbb{R}^6$ be any unit vector and $\boldsymbol{\mathcal{W}}_2(t_0, t_f)$ the observability Grammian associated with the pair $(\boldsymbol{\mathcal{A}}_2(t), \mathbf{C}_2)$. Then,

$$\begin{aligned} \mathbf{c}^T \boldsymbol{\mathcal{W}}_2(t_0, t_f)\,\mathbf{c} &= \int_{t_0}^{t_f} \left\| \mathbf{C}_2(t)\boldsymbol{\phi}_2(t, t_0)\,\mathbf{c} \right\|^2 dt \\ &= \int_{t_0}^{t_f} \left\| \mathbf{f}_2(t, t_0) \right\|^2 dt, \end{aligned}$$

where

$$\mathbf{f}_2(t, t_0) = \mathbf{c}_1 + \left(\int_{t_0}^{t} \mathbf{J}(\sigma)\mathbf{R}^T(\sigma)\mathbf{R}(t_0)\, d\sigma \right) \mathbf{c}_2.$$

To show necessity, suppose that there exists a unit vector $\mathbf{d} \in \mathbb{R}^3$ such that, for all $t \in \mathcal{I}$, $\mathbf{J}(t)\mathbf{R}^T(t)\mathbf{R}(t_0)\mathbf{d} = \mathbf{0}$. Let $\mathbf{c} = \left[\begin{array}{cc} \mathbf{0} & \mathbf{d}^T \end{array} \right]^T$. Then, one directly concludes that $\mathbf{f}_2(t, t_0) = \mathbf{0}$ for all $t \in \mathcal{I}$ and hence $\mathbf{c}^T \boldsymbol{\mathcal{W}}_2(t_0, t_f)\,\mathbf{c} = 0$, which implies that the observability Grammian $\boldsymbol{\mathcal{W}}_2(t_0, t_f)$ is not invertible. This proves necessity.

In turn, to show sufficiency, again, one needs to consider every possible unit vector $\mathbf{c} = \left[\begin{array}{cc} \mathbf{c}_1^T & \mathbf{c}_2^T \end{array} \right]^T$. Consider first the case $\mathbf{c}_1 \neq \mathbf{0}$. In that case, $\|\mathbf{f}_2(t_0, t_0)\| = \|\mathbf{c}_1\| > 0$ and hence, using Lemma 1, there exists $t_1 \in]t_0, t_f]$ such that $\mathbf{c}^T \boldsymbol{\mathcal{W}}_2(t_0, t_1)\,\mathbf{c} > 0$ Suppose now that $\mathbf{c}_1 = \mathbf{0}$. As $\mathbf{c}$ is a unit vector, it follows that in that case $\mathbf{c}_2$ is also a unit vector and $\mathbf{f}_2(t_0, t_0) = \mathbf{0}$. But now

$$\frac{\partial}{\partial t}\mathbf{f}_2(t, t_0) = \mathbf{J}(t)\mathbf{R}^T(t)\mathbf{R}(t_0)\,\mathbf{c}_2.$$

Under the conditions of the proposition, for all unit vectors $\mathbf{c}_2$, it is possible to choose a time instant $t_i \in \mathcal{I}$ such that

$$\left\| \left. \frac{\partial}{\partial t}\mathbf{f}_2(t, t_0) \right|_{t=t_i} \right\| > 0.$$

Then, using Lemma 1 again, it follows that, for all unit vectors $\mathbf{c}$ with $\mathbf{c}_1 = \mathbf{0}$, it is possible to choose $t_j \in]t_0, t_i] \subset]t_0, t_f]$ such that $\mathbf{c}^T \boldsymbol{\mathcal{W}}_2(t_0, t_j)\,\mathbf{c} > 0$. Thus it is shown that, for all unit vectors $\mathbf{c}$ there exists $t_k \in]t_0, t_f]$ such that $\mathbf{c}^T \boldsymbol{\mathcal{W}}_2(t_0, t_k)\,\mathbf{c} > 0$, which implies that, for all unit vectors $\mathbf{c}$, $\mathbf{c}^T \boldsymbol{\mathcal{W}}_2(t_0, t_f)\,\mathbf{c} > 0$. This concludes the proof. ■

Interestingly, uniform complete observability can be established as before. Uniform bounds must be invoked to show that the LTV system (7) is UCO if and only if there exist positive constants $\alpha > 0$ and $\delta > 0$ such that, for all $t \geq t_0$ and all unit vectors $\mathbf{d} \in \mathbb{R}^3$, it is possible to choose $t_i \in [t, t+\delta]$ such that $\left\| \mathbf{J}(t_i)\mathbf{R}^T(t_i)\mathbf{R}(t)\mathbf{d} \right\| \geq \alpha$.

C. Constant body-fixed velocity model with reduced sensor suite

In compact form, it is possible to rewrite (4) as

$$\left\{ \begin{array}{l} \dot{\mathbf{x}}_3(t) = \mathbf{A}_3(t)\mathbf{x}_3(t) + \mathbf{B}_3 u_3(t) \\ \mathbf{y}_3(t) = \mathbf{C}_3\mathbf{x}_3(t) \end{array} \right., \tag{8}$$

where $\mathbf{x}_3(t) = \left[h_x^T(t)\ \mathbf{v}^T(t) \right]^T \in \mathbb{R}^4$ is the system state,

$$u_3(t) = -\omega_z(t)h_y(t) + \omega_y(t)h_z(t) \in \mathbb{R}$$

is the system input,

$$\mathbf{A}_3(t) = \left[\begin{array}{cc} 0 & \mathbf{J}_x(t) \\ \mathbf{0} & \mathbf{0} \end{array} \right] \in \mathbb{R}^{4\times 4},\ \mathbf{B}_3 = \left[\begin{array}{c} 1 \\ \mathbf{0} \end{array} \right] \in \mathbb{R}^4,$$

and $\mathbf{C}_3 = \left[\begin{array}{cc} 1 & \mathbf{0} \end{array} \right] \in \mathbb{R}^{1\times 4}$, which is a linear time-varying system. The following proposition establishes a necessary and sufficient condition on the observability of (8).

Proposition 3: Suppose that, for all unit vectors $\mathbf{d} \in \mathbb{R}^3$, it is possible to choose $t_i \in \mathcal{I}$, $\mathcal{I} := [t_0, t_f]$, such that $\mathbf{J}_x(t_i)\mathbf{d} \neq 0$. Then, the LTV system (6) is observable on $\mathcal{I}$.

Proof: The proof follows similar similar steps to that of Proposition 1 and hence it is omitted. ■

Similarly, uniform complete observability for the LTV system (8) can be established. The LTV system (8) is uniformly completely observable if and only if there exist positive constants $\alpha > 0$ and $\delta > 0$ such that, for all $t \geq t_0$ and all unit vectors $\mathbf{d} \in \mathbb{R}^3$, it is possible to choose $t_i \in [t, t+\delta]$ such that $|\mathbf{J}_x(t_i)\mathbf{d}| \geq \alpha$.

D. Constant inertial velocity model with reduced sensor suite

In compact form, it is possible to rewrite (5) as

$$\left\{ \begin{array}{l} \dot{\mathbf{x}}_4(t) = \mathbf{A}_4(t)\mathbf{x}_4(t) + \mathbf{B}_4 u_4(t) \\ \mathbf{y}_4(t) = \mathbf{C}_4\mathbf{x}_4(t) \end{array} \right., \tag{9}$$

where $\mathbf{x}_4(t) = \left[h_x^T(t)\ \mathbf{v}^T(t) \right]^T \in \mathbb{R}^4$ is the system state,

$$u_4(t) = -\omega_z(t)h_y(t) + \omega_y(t)h_z(t) \in \mathbb{R}$$

is the system input,

$$\mathbf{A}_4(t) = \left[\begin{array}{cc} 0 & \mathbf{J}_x(t) \\ \mathbf{0} & -\mathbf{S}(\boldsymbol{\omega}(t)) \end{array} \right] \in \mathbb{R}^{4\times 4},\ \mathbf{B}_4 = \left[\begin{array}{c} 1 \\ \mathbf{0} \end{array} \right] \in \mathbb{R}^4,$$

and $\mathbf{C}_4 = \left[\begin{array}{cc} 1 & \mathbf{0} \end{array} \right] \in \mathbb{R}^{1\times 4}$, which is a linear time-varying system. The following proposition establishes a necessary and sufficient condition on the observability of (9).

Proposition 4: Suppose that, for all unit vectors $\mathbf{d} \in \mathbb{R}^3$, it is possible to choose $t_i \in \mathcal{I}$, $\mathcal{I} := [t_0, t_f]$, such that $\mathbf{J}_x(t_i)\mathbf{R}^T(t_i)\mathbf{R}(t_0)\mathbf{d} \neq 0$. Then, the LTV system (9) is observable on $\mathcal{I}$.

Proof: The proof follows similar similar steps to that of Proposition 2 and hence it is omitted. ■

Again, one can show that the LTV system (8) is UCO if and only if there exist positive constants $\alpha > 0$ and $\delta > 0$ such that, for all $t \geq t_0$ and all unit vectors $\mathbf{d} \in \mathbb{R}^3$, it is possible to choose $t_i \in [t, t+\delta]$ such that $\left| \mathbf{J}_x(t_i)\mathbf{R}^T(t_i)\mathbf{R}(t)\mathbf{d} \right| \geq \alpha$.

IV. A SIMPLE CASE STUDY

The previous results allows one to relax the initial assumption of all times non-singularity of the Jacobian $\mathbf{J}(t)$. Instead, a low-rank Jacobian can be considered provided that the non-null space maps the whole space over a finite time interval. This is a simple rephrasing of the previous statements. This opens new perspectives of both practical and theoretical nature. To illustrate this, we now use some very simple examples.

a) Full turns: Consider the following simple system, in a 2-dimensional space, in which a rigid body equipped with a magneto-inertial system is traveling in an area where the magnetic field has a uniform (constant) Jacobian. In the $\{I\}$ inertial frame of reference, the particular motion that is considered is

$$\mathbf{h}\left(\mathbf{p}(t)\right) = \begin{bmatrix} \lambda x(t) \\ 0 \end{bmatrix}, \quad \lambda \neq 0,$$

where

$$\mathbf{p}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

denotes the position of the device at time t. In this particular case, the x direction of $\{I\}$ is chosen aligned with the direction of non-zero variation of the magnetic field $\mathbf{h}$. A constant field could be added for increased generality (see last paragraph of this section) but does not bring anything to nor harms the observability properties, as it leaves unchanged the equations governing the variations of the sensed field coupled to the motion of the device. Further, it is assumed that the trajectory is described by the following pseudo-periodic equations

$$\mathbf{p}(t) = \begin{bmatrix} \gamma_1 t + \gamma_2 \sin(\omega t) \\ -\gamma_2 \cos(\omega t) \end{bmatrix}$$

and that the angle between the body frame of reference $\{B\}$ and $\{I\}$ is simply ωt, $w > 0$ such that the rotation matrix between these two frames is

$$\mathbf{R}(t) = \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{bmatrix}.$$

Note that the body frame is not aligned with the trajectory at all times. The Jacobian of the magnetic field $\mathbf{h}$, in inertial coordinates, is simply

$$\mathbf{J}_i = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}. \tag{10}$$

Once it is projected onto the body-fixed reference frame, it gives

$$\begin{aligned} \mathbf{J}(t) &= \mathbf{R}(t)^T \mathbf{J}_i \mathbf{R}(t) \\ &= \begin{bmatrix} \lambda \cos(\omega t)^2 & -\lambda \cos(\omega t)\sin(\omega t) \\ -\lambda \sin(\omega t)\cos(\omega t) & \lambda \sin(\omega t)^2 \end{bmatrix}, \end{aligned}$$

which is obviously of rank 1, as $\mathbf{J}_i$ is.

Let $\mathbf{d}_0 = [1 \quad 0]^T$. It is a unit vector such that $\mathbf{J}_i \mathbf{d}_0 \neq \mathbf{0}$. Under the assumption $\omega > 0$, the set of matrices $\{t \in [0, 2\pi/\omega], \mathbf{R}(t)\}$ can be used to map any unit vector of $\mathbb{R}^2$ onto $\mathbf{d}_0$. For any unit vector $\mathbf{d}$, choose t_i such that $\mathbf{d}_0 = \mathbf{R}\left(t_i\right)\mathbf{d}$. Then, if follows that $\mathbf{R}\left(t_i\right)^T \mathbf{J}_i \mathbf{R}\left(t_i\right)\mathbf{d} \neq \mathbf{0}$, which permits to conclude on the observability of the dynamics thanks to Proposition 1. This shows that observability has been attained in this singular case thanks to the rotating nature of the trajectory.

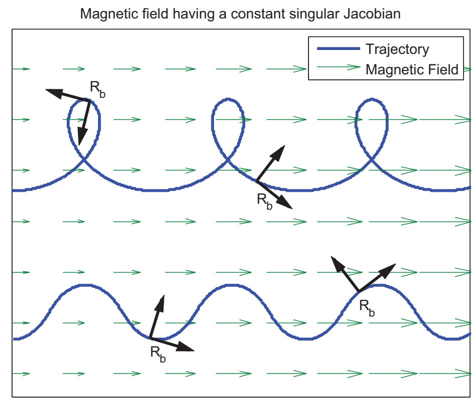

Fig. 1. A spinning trajectory is used to recover obervability in an area where the magnetic field has a constant Jacobian.

b) Partial turns: Interestingly, it is not necessary that the body reference of the device makes full turns. This possibility is pictured in Figure 1. Consider now that the motion of the rigid body is generated by

$$\dot{\mathbf{p}}(t) = \gamma_1 \begin{bmatrix} \cos\alpha(t) \\ \sin\alpha(t) \end{bmatrix}, \quad \alpha(t) = \gamma_2 \sin(\gamma_3 t),$$

with $\gamma_1, \gamma_2, \gamma_3$ non-zero constant parameters. One such trajectory for $\gamma_2 = 1$ is pictured in Fig. 1. One has then

$$\mathbf{R}(t) = \begin{bmatrix} \cos\alpha(t) & -\sin\alpha(t) \\ \sin\alpha(t) & \cos\alpha(t) \end{bmatrix}.$$

Let $\mathbf{d}_1 = [0 \quad 1]^T$ denote one unit vector that spans the null-space of $\mathbf{J}_i$. Consider again the assumption of Proposition 1. For any unit vector $\mathbf{d}$ either $\mathbf{J}_i \mathbf{R}\left(t_0\right)\mathbf{d} \neq \mathbf{0}$, which means that $\mathbf{d}$ satisfies the assumption of the Proposition with $t_i = t_0$, or $\mathbf{J}_i \mathbf{R}\left(t_0\right)\mathbf{d} = \mathbf{0}$, which equivalently means $\left(\mathbf{R}\left(t_0\right)\mathbf{d}\right) \times \mathbf{d}_1 = 0$. Consider that $\mathbf{g}\left(t_0\right) := \mathbf{J}_i \mathbf{R}\left(t_0\right)\mathbf{d} = \mathbf{0}$. Differentiating this expression with respect to time t_0 gives

$$\dot{\mathbf{g}}\left(t_0\right) = \mathbf{J}_i \dot{\alpha}\left(t_0\right) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{R}\left(t_0\right)\mathbf{d}.$$

If $\dot{\alpha}(t_0) = 0$, then $t_0 = \pi/2 \mod \pi$, then $\mathbf{J}_i\left(t_0\right)\mathbf{R}\left(t_0\right)\mathbf{d} \neq \mathbf{0}$, which is a contradiction. Therefore, $\dot{\alpha}(t_0) \neq 0$, and thus

$$\dot{\mathbf{g}}\left(t_0\right) = \left(\dot{\alpha}(t_0) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{R}\left(t_0\right)\mathbf{d}\right) \times \mathbf{d}_1 \neq \mathbf{0}.$$

Therefore, there exists $t_1 > t_0$ such that $\mathbf{R}\left(t_1\right)^T \mathbf{J}_i \mathbf{R}\left(t_1\right)\mathbf{d} \neq \mathbf{0}$. This gives the conclusion. The systems is observable.

c) Reconstruction with a Kalman filter: To illustrate the practical importance of the relaxation of the assumptions on MINAV, we now consider a test-case of velocity reconstruction with a Kalman filter. Without loss of generality [2], the magnetic field that can be measured is now

$$\mathbf{h}\left(\mathbf{p}(t)\right) = \begin{bmatrix} 1 + \lambda x(t) \\ 2 \end{bmatrix}, \quad \lambda \neq 0,$$

where $\mathbf{p}$ satisfies the equation of motion

$$\dot{\mathbf{p}}(t) = \mathbf{R}(t) \begin{pmatrix} \gamma_1 \\ 0 \end{pmatrix}$$

[2] constant values have been added to the sensed magnetic field, as previously announced

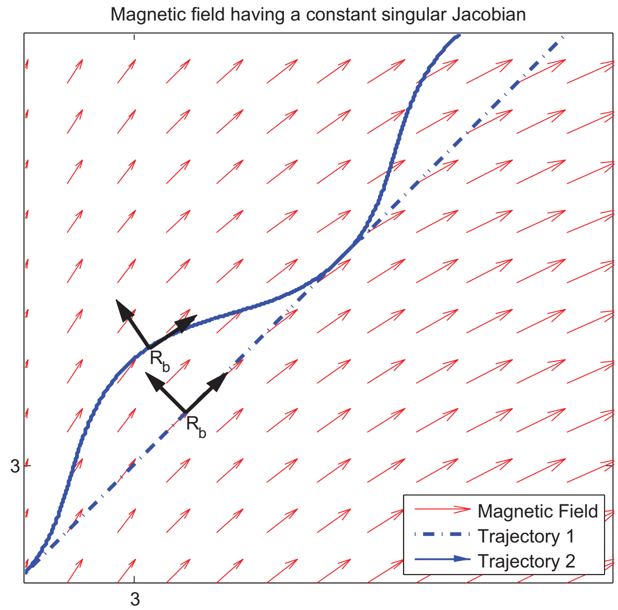

Fig. 2. Two trajectories going though an area where the magnetic field has a constant Jacobian.

and

$$\alpha(t) = \gamma_2 \sin(\gamma_3 t).$$

as before. The Jacobian of the magnetic field is given by (10) and is singular. According to the previous discussion, we know that the system is observable for $\gamma_3 \neq 0$. In Figure 2 two trajectories are pictured, for $\gamma_3 = \frac{1}{2}$ (curly path) and $\gamma_3 = 0$ (straight line), respectively. Uniform Complete Observability which can be established here for $\gamma_3 > 0$, guarantees convergence of the Kalman filter. This point is illustrated in Figure 3. As the rigid body travels into the magnetically perturbed area, the Kalman filter manages to reconstruct the unknown velocity in the case $\gamma_3 \neq 0$. Otherwise, the Kalman filter asymptotically reaches biased values.

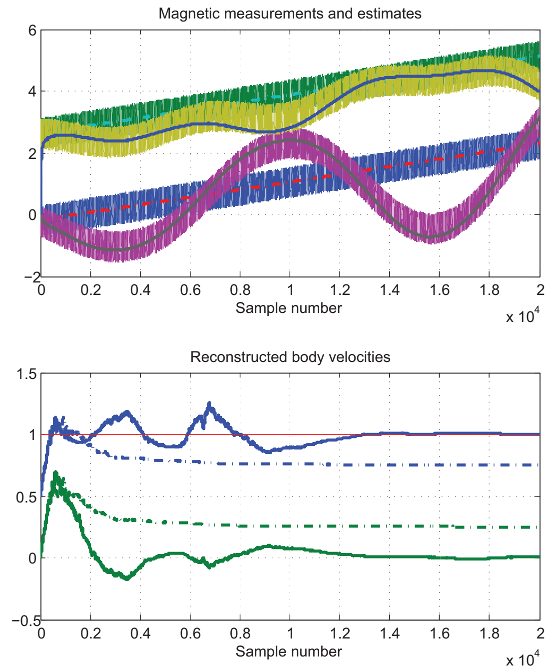

Fig. 3. Reconstruction with a Kalman filter, of the systems states for the two trajectories of Figure 2. Above: magnetic field measurements and reconstructions. Below: Reconstruction of the velocities. The true velocities $[\gamma_1 = 1, 0]^T$ are reconstructed when the spinning parameter γ_3 is non-zero, thanks to the recovered observability. The case $\gamma_3 = 0$ yields biased estimates.

V. Conclusions

In this paper, we have proven that the classic assumption formulated for the application of MINAV technique was unnecessarily restrictive. A careful study of observability in the non-stationary case reveals that mild assumptions on the nature of the trajectory followed by the sensing system are sufficient to obtain convergence of state observers. This is certainly an important and enlightening point to interpret experimental results. Further studies could be focused on more advanced case studies (in 3-dimensional spaces) among other possibilities.

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