2013 21st Mediterranean Conference on Control & Automation (MED)
Platanias-Chania, Crete, Greece, June 25-28, 2013

GAS Decentralized Navigation Filters in a Continuous-Discrete Fixed Topology Framework

Daniel Viegas, Pedro Batista, Paulo Oliveira, and Carlos Silvestre

Abstract— **This paper addresses the problem of state estimation in formations of autonomous vehicles. The approach considered here consists in the implementation of a local state observer in each vehicle relying only on locally available measurements and data communicated by neighboring agents, resulting in a decentralized state observer which features lower computational and communication loads than comparable centralized solutions. A method for computing observer gains which yield globally asymptotically stable error dynamics is presented for fixed topology formations, as well as an iterative algorithm for improving the decentralized estimator's performance when the measurements are corrupted by noise. The proposed framework is particularized to the practical case of a formation of Autonomous Underwater Vehicles (AUVs), and a continuous-discrete formulation is achieved for the local state observers, to take into account the difference in sampling rates between on-board instrumentation and positioning systems. To assess the performance of the solution, simulation results are presented and discussed for different formation topologies.**

I. Introduction

Motivated by the wealth of potential applications for formations composed by multiple agents working cooperatively, see e.g. [5], [9], and [12], the subjects of estimation and control in formations of vehicles have been researched extensively in the past few years, yielding many compelling contributions. Centralized solutions consider the formation of vehicles as a whole, usually relying on a central processing node to execute most, if not all, computations. This approach is attractive from a design point of view as most classical, single-agent solutions can still be applied. On the flip side, implementation is almost guaranteed to be cumbersome, as heavy computational and communication loads are to be expected due to the necessity of conveying all the information in the formation to a central processing node, which must then relay the results of its computations to the vehicles. To avoid those pitfalls, the aim of decentralized solutions is to break down the problem in several parts, leaving each agent in the formation with the responsibility of performing a subset of the computations, relying on limited information and communication with other vehicles in the formation. However, there is a trade-off: while implementation is simpler and more efficient, design and analysis become much more intricate, see e.g. [6] and [21].

This work was partially supported by Fundação para a Ciência e a Tecnologia (FCT) under Project [PEst-OE/EEI/LA0009/2011], by the EU Project TRIDENT (Contract No. 248497), and by the FCT Project PTDC/EEACRO/111197/2009 MAST/AM. The work of D. Viegas was supported by the PhD Scholarship SFRH/BD/71486/2010 from FCT.

D. Viegas, P. Batista, and P. Oliveira are with the Institute for Systems and Robotics, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisboa, Portugal. C. Silvestre is with the Institute for Systems and Robotics and the Department of Electrical and Computer Engineering, Faculty of Science and Technology, University of Macau, Taipa, Macau. {dviegas,pbatista,pjcro,cjs}@isr.ist.utl.pt

On the subject of decentralized state estimation, interesting approaches can be found in [17], [22], and [2]. The closely related area of decentralized control has also seen a wealth of relevant solutions, such as in [10] and [18]. Most notably, recent work on the subject of quadratic invariance, see e.g. [13] and [15], has offered optimal solutions for certain classes of formations. In contrast, the work presented here focuses on methods which can be applied to any formation topology, trading off optimality for generality.

This paper addresses the problem of decentralized state estimation of linear motion quantities in formations of vehicles. In the envisioned scenario, each agent in the formation aims to estimate its own position and velocity based on some awareness of its own movement and local measurements and communication. Each agent is assumed to have access to either measurements of its own state, or measurements of its state relative to one or more vehicles in the vicinity, as well as the states estimates of those agents, received through communication. A method for local state observer design is presented for acyclic formations, featuring globally asymptotically stable (GAS) estimation error dynamics. Building on this, an iterative algorithm is presented for improving the performance of the decentralized state estimator in noisy environments. The algorithm aims to minimize the H_2 norm of the estimation error dynamics, and is applicable to any fixed formation topology. The proposed framework is then particularized to the practical case of a formation of Autonomous Underwater Vehicles (AUVs), and an equivalent continuous-discrete formulation for local observer design is introduced to take advantage of the much faster sampling rate of on-board sensors in comparison with the positioning system. Previous work by the authors on this topic can be found in [19]. This paper extends those results to the discrete-time framework and explores the application to AUVs in greater depth.

The rest of the paper is organized as follows: Section II introduces the dynamics of the vehicles and also of the local observers, while Section III reports the stability and performance analysis of the decentralized state estimator. Section IV details the aforementioned application to a formation of AUVs, and Section V shows simulation results. Finally, Section VI summarizes the main conclusions of the paper.

A. Notation

Throughout the paper the symbol $\mathbf{0}$ denotes a matrix (or vector) of zeros and $\mathbf{I}$ an identity matrix, both of appropriate dimensions. Whenever relevant, the dimensions of an $n \times n$ identity matrix are indicated as $\mathbf{I}_n$. A block diagonal matrix is represented as $\operatorname{diag}(\mathbf{A}_1, \ldots, \mathbf{A}_n)$, and the Kronecker product of two matrices $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathbf{A} \otimes \mathbf{B}$. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, $\mathbf{x} \times \mathbf{y}$ represents the cross product.

II. Singe Vehicle Kinematics and Local State Observer Design

Consider a formation composed by N autonomous vehicles moving in a scenario, where each vehicle is identified by a distinct positive integer $i \in \{1, 2, \ldots, N\}$. It is assumed that the topology of the formation is fixed, in the sense that the measurements and communication links available to each agent do not change over the course of the mission. The state $\mathbf{x}_i(t) \in \mathbb{R}^{n_L}$ of vehicle i, to be estimated, follows

$$\dot{\mathbf{x}}_i(t) = \mathbf{A}_L \mathbf{x}_i(t) + \mathbf{B}_L \mathbf{u}_i(t),$$

where $\mathbf{u}_i(t) \in \mathbb{R}^{m_L}$ is the input of the system, and $\mathbf{A}_L \in \mathbb{R}^{n_L \times n_L}$ and $\mathbf{B}_L \in \mathbb{R}^{n_L \times m_L}$ are given constant matrices.

Regarding the available measurements, suppose that one or more vehicles have access to measurements of their own state, denoted as "absolute" measurements for convenience, yielding the Linear Time-Invariant (LTI) system

$$\begin{cases} \dot{\mathbf{x}}_i(t) = \mathbf{A}_L \mathbf{x}_i(t) + \mathbf{B}_L \mathbf{u}_i(t) \\ \mathbf{y}_i(t) = \mathbf{C}_L \mathbf{x}_i(t) \end{cases}, \tag{1}$$

where $\mathbf{y}_i(t) \in \mathbb{R}^{o_L}$ is the output of the system, and $\mathbf{C}_L \in \mathbb{R}^{o_L \times n_L}$. However, many commonly used positioning systems such as Ultra-short Baseline (USBL), Long Baseline (LBL), and the GPS provide samples at a relatively slow rate, and as such the continuous-time framework is not the most suitable for observer design. Suppose that the output $\mathbf{y}_i(t)$ is sampled with constant sampling period T. Then, the system (1) can be described by the following discrete-time representation for observer design purposes:

$$\begin{cases} \mathbf{x}_i(k+1) = \mathbf{A}_D \mathbf{x}_i(k) + \mathbf{u}_i^D(t) \\ \mathbf{y}_i(k) = \mathbf{C}_D \mathbf{x}_i(k) \end{cases}, \tag{2}$$

where $\mathbf{A}_D = e^{\mathbf{A}_L T}$, $\mathbf{C}_D = \mathbf{C}_L$, and

$$\mathbf{u}_i^D(t) = \int_0^T e^{\mathbf{A}_L (T-\tau)} \mathbf{B}_L \mathbf{u}_i(t_k + \tau) d\tau,$$

in which t_k denotes the sampling instant of sample k. It is assumed that the pair $(\mathbf{A}_D, \mathbf{C}_D)$ is observable.

For the other vehicles, suppose that each one has access to measurements of its state relative to N_i other vehicles in the vicinity, yielding the dynamic system

$$\begin{cases} \dot{\mathbf{x}}_i(t) = \mathbf{A}_L \mathbf{x}_i(t) + \mathbf{B}_L \mathbf{u}_i(t) \\ \mathbf{y}_i(t) = \mathbf{C}_i \Delta \mathbf{x}_i(t) \end{cases}, \tag{3}$$

in which $\mathbf{y}_i(t) \in \mathbb{R}^{o_L N_i}$, $\mathbf{C}_i = \mathbf{I}_{N_i} \otimes \mathbf{C}_L$, and

$$\Delta \mathbf{x}_i(t) := \begin{bmatrix} \mathbf{x}_i(t) - \mathbf{x}_{\theta_{i,1}}(t) \\ \mathbf{x}_i(t) - \mathbf{x}_{\theta_{i,2}}(t) \\ \vdots \\ \mathbf{x}_i(t) - \mathbf{x}_{\theta_{i,N_i}}(t) \end{bmatrix} \in \mathbb{R}^{n_L N_i}, \ \theta_{i,j} \in \Theta_i,$$

where

$$\Theta_i := \{\theta_{i,1}, \theta_{i,2}, \ldots, \theta_{i,N_i} \,|\, \theta_{i,j} \in \{1, \ldots, N\}, \, j = 1, \ldots, N_i\}$$

is the set of other vehicles corresponding to the relative measurements available to vehicle i. Furthermore, assume that those vehicles send updated state estimates to vehicle i through communication. Supposing that the output is sampled with constant sampling period T, as in the previous case, the following discrete-time representation can be achieved:

$$\begin{cases} \mathbf{x}_i(k+1) = \mathbf{A}_D \mathbf{x}_i(k) + \mathbf{u}_i^D(k) \\ \mathbf{y}_i(k) = \mathbf{C}_i^D \Delta \mathbf{x}_i(k) \end{cases}, \tag{4}$$

with $\mathbf{C}_i^D = \mathbf{C}_i$.

A. Local state observer design

For the vehicles which have access to absolute measurements, since the pair $(\mathbf{A}_D, \mathbf{C}_D)$ is observable, it is straightforward to design a local state observer with GAS error dynamics for the LTI system (2), see [1]. For vehicle i, its dynamics follow

$$\begin{cases} \hat{\mathbf{x}}_i(k+1) = \mathbf{A}_D \hat{\mathbf{x}}_i(k) + \mathbf{u}_i^D(t) + \mathbf{L}_i(\mathbf{y}_i(k) - \hat{\mathbf{y}}_i(k)) \\ \hat{\mathbf{y}}_i(k) = \mathbf{C}_D \hat{\mathbf{x}}_i(k) \end{cases},$$

where $\hat{\mathbf{x}}_i(k) \in \mathbb{R}^{n_L}$ is the state estimate, and $\mathbf{L}_i \in \mathbb{R}^{n_L \times o_L}$ is a constant matrix of observer gains, to be computed.

Regarding the vehicles which have access to relative measurements, the design process is slightly different. First, build an estimate of $\Delta \mathbf{x}_i(k)$ using the state estimates received through communication,

$$\Delta \hat{\mathbf{x}}_i(k) := \begin{bmatrix} \hat{\mathbf{x}}_i(k) - \hat{\mathbf{x}}_{\theta_{i,1}}(k) \\ \hat{\mathbf{x}}_i(k) - \hat{\mathbf{x}}_{\theta_{i,2}}(k) \\ \vdots \\ \hat{\mathbf{x}}_i(k) - \hat{\mathbf{x}}_{\theta_{i,N_i}}(k) \end{bmatrix} \in \mathbb{R}^{n_L N_i}, \ \theta_{i,j} \in \Theta_i.$$

The local observer structure for the system (4) follows

$$\begin{cases} \hat{\mathbf{x}}_i(k+1) = \mathbf{A}_L \hat{\mathbf{x}}_i(k) + \mathbf{u}_D^i(k) + \mathbf{L}_i(\mathbf{y}_i(k) - \hat{\mathbf{y}}_i(k)) \\ \hat{\mathbf{y}}_i(k) = \mathbf{C}_i^D \Delta \hat{\mathbf{x}}_i(k) \end{cases}, \tag{5}$$

with $\mathbf{L}_i \in \mathbb{R}^{n_L \times o_L N_i}$.

III. Stability and Performance of the Decentralized State Observer

As formations such as the ones considered in this paper can be conveniently described by a directed graph, it is convenient to introduce a few concepts of graph theory before proceeding. A graph $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ consists in a set $\mathcal{V}$ of vertices along with a set $\mathcal{E}$ of edges. In a directed graph, or digraph, an edge is composed by an ordered pair of vertices $e = (a, b)$. In this paper, this notation means that edge e is incident on a and b, and directed towards b. A directed path in $\mathcal{G}$ is a sequence $(v_0, e_1, v_1, e_2, v_2, \ldots, e_n, v_n)$ of vertices and edges of $\mathcal{G}$ such that $e_i = (v_{i-1}, v_i)$, in which all vertices except the first and the last must be different, and a directed cycle is a directed path with the added restriction that the first and last vertices are the same. A directed graph is acyclic if there are no directed cycles in it. The kind of formation that was detailed in the previous section can be represented by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, in which each vertex represents a distinct vehicle, and an edge (a, b) means that vehicle b has access to a measurement relative to vehicle a, and also to its state estimate. Examples of such formation graphs are depicted in Fig. 1. In order to incorporate the absolute measurements available to some of the vehicles, it is convenient introduce an additional set of edges of the form $(0, i)$, connected to only one vertex. For a graph $\mathcal{G}$ with n_v vertices and n_e edges, the entries of its incidence matrix $\mathbf{S}_\mathcal{G} \in \mathbb{R}^{n_v \times n_e}$ follow

$$[\mathbf{S}_\mathcal{G}]_{jk} = \begin{cases} 1, & \text{edge } k \text{ incident on } j \text{, directed towards it,} \\ -1, & \text{edge } k \text{ incident on } j \text{, directed away from it,} \\ 0, & \text{edge } k \text{ not incident on } j. \end{cases}$$

For a more in-depth treatment of the concepts summarized above, see e.g. [20]. The following result establishes a sufficient condition for global asymptotic stability of the estimation error for the distributed state observer.

Theorem 1: Consider a formation composed of N agents, whose dynamics are described either by (2) or (4), depending on the type of measurements available to them, and assume that the digraph associated with the formation is acyclic. Suppose that each agent i described by (2) implements a local state observer with GAS error dynamics, with gain $\mathbf{L}_i \in \mathbb{R}^{n_L \times o_L}$, and that each agent j described by (4) implements the local state observer (5), with the gain $\mathbf{L}_j$ chosen so that the matrix $(\mathbf{A}_D - \mathbf{L}_j \mathbf{C}_i^D)$ is discrete-time stable. Let $\tilde{\mathbf{x}}_i(k) := \mathbf{x}_i(k) - \hat{\mathbf{x}}_i(k) \in \mathbb{R}^{n_L}$ denote the estimation error of the local observer at vehicle i. Then, the estimation error of the distributed state observer, $\tilde{\boldsymbol{x}}(k) := \begin{bmatrix}\tilde{\mathbf{x}}_1^T(k) & \tilde{\mathbf{x}}_2^T(k) & \dots & \tilde{\mathbf{x}}_N^T(k)\end{bmatrix}^T \in \mathbb{R}^{n_L N}$, composed by the concatenation of the estimation error of each local observer, converges globally asymptotically to zero, and its dynamics satisfy $\tilde{\boldsymbol{x}}(k+1) = \boldsymbol{\Lambda}\tilde{\boldsymbol{x}}(k)$ for some $\boldsymbol{\Lambda} \in \mathbb{R}^{n_L N \times n_L N}$, whose eigenvalues are those of each local state observer.

The proof is analogous to its continuous-time counterpart, as described in [19], and is omitted due to space constraints. This result allows the design of a distributed estimator in the terms described in Section II. Note that the state observer of each agent can be designed locally and results from the solution of simple stable pole placement problems.

A. Formation-wide dynamics

To study and improve the performance of the decentralized state observer, it is necessary to consider the global dynamics of the formation, which can be represented in the LTI form

$$\begin{cases} \boldsymbol{x}(k+1) = \mathbf{A}_g \boldsymbol{x}(k) + \boldsymbol{u}(k) + \mathbf{w}(k) \\ \boldsymbol{y}(k) = \mathbf{C}_g \boldsymbol{x}(k) + \mathbf{v}(k) \end{cases}, \tag{6}$$

where $\boldsymbol{x}(k) := \begin{bmatrix}\mathbf{x}_1^T(k) & \dots & \mathbf{x}_N^T(k)\end{bmatrix}^T \in \mathbb{R}^{n_L N}$ is the state of the whole formation, $\boldsymbol{y}(k) := \begin{bmatrix}\mathbf{y}_1^T(k) & \dots & \mathbf{y}_N^T(k)\end{bmatrix}^T \in \mathbb{R}^{o_L M}$ the output of the system, M being the total number of absolute and relative position measurements in the whole formation, and $\boldsymbol{u}(k) := \begin{bmatrix}(\mathbf{u}_1^D)^T(k) & \dots & (\mathbf{u}_N^D)^T(k)\end{bmatrix}^T \in \mathbb{R}^{m_L N}$ is the input of the system. The variables $\mathbf{w}(k) \in \mathbb{R}^{n_L N}$ and $\mathbf{v}(k) \in \mathbb{R}^{o_L M}$ represent, respectively, process and observation noise, which are assumed to be zero-mean uncorrelated white Gaussian processes, with associated covariance matrices $\boldsymbol{\Xi} \in \mathbb{R}^{n_L N \times n_L N}$ and $\boldsymbol{\Theta} \in \mathbb{R}^{o_L M \times o_L M}$. The matrices $\mathbf{A}_g \in \mathbb{R}^{n_L N \times n_L N}$ and $\mathbf{C}_g \in \mathbb{R}^{o_L M \times n_L N}$ are built from the dynamics of the individual agents, following

$$\begin{cases} \mathbf{A}_g = \mathbf{I}_N \otimes \mathbf{A}_D \\ \mathbf{C}_g = \mathbf{S}_{\mathcal{G}}^T \otimes \mathbf{C}_D \end{cases}.$$

The local state observers can also be grouped, yielding

$$\begin{cases} \hat{\boldsymbol{x}}(k+1) := \mathbf{A}_g \hat{\boldsymbol{x}}(k) + \boldsymbol{u}(k) + \mathbf{L}(\boldsymbol{y}(k) - \hat{\boldsymbol{y}}(k)) \\ \hat{\boldsymbol{y}}(k) := \mathbf{C}_g \hat{\boldsymbol{x}}(k) \end{cases}, \tag{7}$$

where $\hat{\boldsymbol{x}}(k) := \begin{bmatrix}\hat{\mathbf{x}}_1^T(k) & \hat{\mathbf{x}}_2^T(k) & \dots & \hat{\mathbf{x}}_N^T(k)\end{bmatrix}^T \in \mathbb{R}^{n_L N}$ is the global state estimate of the decentralized state observer, and $\mathbf{L} \in \mathbb{R}^{n_L N \times o_L M}$ is the matrix of observer gains. To account for the fact that each local observer only has access to some measurements, $\mathbf{L}$ must follow a special structure, or sparsity constraint. More specifically, define an augmented incidence matrix, $\mathbf{S}'_{\mathcal{G}} = S_{\mathcal{G}} \otimes \mathbf{1}_{n,3} \in \mathbb{R}^{n_L N \times o_L M}$, where $\mathbf{1}_{n,m}$ is a $n \times m$ matrix whose entries are all equal to 1. Then, the individual entries of $\mathbf{L}$ follow

$$\begin{cases} [\mathbf{S}'_{\mathcal{G}}]_{ij} = 1 \Rightarrow \mathbf{L}_{ij} \text{ can be set to an arbitrary value,} \\ [\mathbf{S}'_{\mathcal{G}}]_{ij} \neq 1 \Rightarrow \mathbf{L}_{ij} = 0. \end{cases}$$

This can be expressed as a linear constraint for optimization:

$$[\mathbf{L}]_{ij} = 0 \quad \text{if} \quad [\mathbf{S}'_{\mathcal{G}}]_{ij} \neq 1, \quad \forall i \in \{1, 2, \dots, nN\},\ j \in \{1, 2, \dots, 3M\}. \tag{8}$$

This constraint prevents the use of classical filter design techniques such as the Kalman filter, and as such a different strategy must be pursued to find suitable observer gains.

B. H_2 Nominal Performance

Consider the discrete-time system

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \\ \mathbf{z}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k) \end{cases}, \tag{9}$$

where $\mathbf{x}(t) \in \mathbb{R}^m$ is the state of the system, $\mathbf{u}(t) \in \mathbb{R}^o$ the input, and $\mathbf{z}(t) \in \mathbb{R}^p$ is the output. The matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are constant real matrices of appropriate dimensions. The H_2 norm of the system can be used as a performance metric for state observers. In fact, when the components of the input $\mathbf{u}(k)$ are independent zero-mean, white Gaussian noise processes, the H_2 norm of the system is also the asymptotic output variance of the system [16]. Using (6) and (7) the global error of the decentralized state observer (7), $\tilde{\boldsymbol{x}}(k)$, can be shown to follow

$$\tilde{\boldsymbol{x}}(k+1) = (\mathbf{A}_g - \mathbf{L}\mathbf{C}_g)\tilde{\boldsymbol{x}}(k) + \mathbf{w}(k) - \mathbf{L}\mathbf{v}(k). \tag{10}$$

Define a zero-mean, uncorrelated, white Gaussian noise process $\mathbf{q}(k) \in \mathbb{R}^{n_L N + o_L M}$ whose covariance is the identity matrix. The error dynamics (10) can then be rewritten as

$$\tilde{\boldsymbol{x}}(k+1) = (\mathbf{A}_g - \mathbf{L}\mathbf{C}_g)\tilde{\boldsymbol{x}}(k) + \begin{bmatrix}\boldsymbol{\Xi}^{\frac{1}{2}} & -\mathbf{L}\boldsymbol{\Theta}^{\frac{1}{2}}\end{bmatrix}\mathbf{q}(k).$$

By making the substitution

$$\begin{cases} \mathbf{A} = \mathbf{A}_g - \mathbf{L}\mathbf{C}_g \\ \mathbf{B} = \begin{bmatrix}\boldsymbol{\Xi}^{\frac{1}{2}} & -\mathbf{L}\boldsymbol{\Theta}^{\frac{1}{2}}\end{bmatrix} \\ \mathbf{C} = \mathbf{I} \\ \mathbf{D} = \mathbf{0} \\ \mathbf{x}(k) = \tilde{\boldsymbol{x}}(k) \\ \mathbf{u}(k) = \mathbf{q}(k) \end{cases}, \tag{11}$$

the system (9) describes the error dynamics of the decentralized state observer, and its H_2 norm is also the asymptotic variance of the estimation error. Thus, the problem of optimizing the performance of the state observer in noisy environments can be restated as minimizing the H_2 norm of (9), where the system variables are given by (11). Define

$$\mathbf{X}(\mathbf{P}, \mathbf{L}) := \begin{bmatrix} \mathbf{P} & (\mathbf{A}_g - \mathbf{L}\mathbf{C}_g)\mathbf{P} & \begin{bmatrix}\boldsymbol{\Xi}^{\frac{1}{2}} & -\mathbf{L}\boldsymbol{\Theta}^{\frac{1}{2}}\end{bmatrix} \\ \mathbf{P}(\mathbf{A}_g - \mathbf{L}\mathbf{C}_g)^T & \mathbf{P} & \mathbf{0} \\ \begin{bmatrix}\boldsymbol{\Xi}^{\frac{1}{2}} & -\mathbf{L}\boldsymbol{\Theta}^{\frac{1}{2}}\end{bmatrix}^T & \mathbf{0} & \mathbf{I} \end{bmatrix}.$$

Using the substitution (11), the minimization of the H_2 norm of (9) considering the constraints (8) imposed by the graph topology can be done solving the optimization problem [16]

$$\begin{aligned} \min_{\substack{\mathbf{P} \in \mathbb{R}^{n_L N \times n_L N} \\ \mathbf{L} \in \mathbb{R}^{n_L N \times o_L M} \\ \mathbf{W} \in \mathbb{R}^{o_L M \times o_L M} \\ \mu \in \mathbb{R}^+}} \quad & \mu \\ \text{subject to:} \quad & \mathbf{X}(\mathbf{P}, \mathbf{L}) \succ 0, \\ & \begin{bmatrix}\mathbf{W} & \mathbf{P} \\ \mathbf{P} & \mathbf{P}\end{bmatrix} \succ \mathbf{0}, \\ & \operatorname{trace}(\mathbf{W}) < \mu, \\ & \text{and } [\mathbf{L}]_{ij} = 0 \text{ if } [\mathbf{S}'_{\mathcal{G}}]_{ij} \neq 1, \\ & \forall i \in \{1, 2, \dots, nN\}, j \in \{1, 2, \dots, 3M\}. \end{aligned} \tag{12}$$

TABLE I

ALGORITHM FOR H_2 NORM MINIMIZATION

1)	Initialization: set $n = 1$; find $\mathbf{L}^{(0)}$ such that $(\mathbf{A}_g - \mathbf{L}^{(0)}\mathbf{C}_g)$ is discrete time stable (this can be done following, e.g., Theorem 1); choose a stopping criterion for the algorithm (e.g. a fixed number of steps, or a minimum improvement on μ at each iteration).
2)	Solve the optimization problem (12) with $\mathbf{L}$ fixed as $\mathbf{L}^{(n-1)}$ and store the resulting $\mathbf{P}^*$ as $\mathbf{P}^{(n)}$.
3)	Solve the optimization problem (12) with $\mathbf{P}$ fixed as $\mathbf{P}^{(n)}$ and store the resulting $\mathbf{L}^*$ as $\mathbf{L}^{(n)}$.
4)	If the stopping criterion is met, stop and take $\mathbf{L}^{(n)}$ as the gain. Otherwise, set $n = n + 1$ and go to step 2.

The resulting set of constraints contains a bilinear matrix inequality (BMI), which is inherently difficult to treat and is usually associated with nonconvex problems. However, if the value of $\mathbf{L}$ is fixed, the constraints take a Linear Matrix Inequality (LMI) form, and there exist very fast and efficient methods to solve optimization problems with LMI constraints. Following this, Table I details an algorithm for improvement of the performance of the decentralized state observer, similar to the $\mathcal{P}-\mathcal{K}$ iterations used in some cases for controller design via BMIs, see e.g. [8] and [11].

IV. APPLICATION TO A FORMATION OF AUVS

Consider a formation composed by N AUVs, and suppose that each has sensors mounted on-board which give access to either measurements of its own position in an inertial reference coordinate frame $\{I\}$, or measurements of its position relative to one or more AUVs in the vicinity. Furthermore, each of those vehicles transmits an estimate of its own inertial position to AUV i. In underwater applications, the relative measurements can be provided by an USBL positioning system in an inverted configuration. The USBL is composed of a small calibrated array of acoustic receivers and measures the distance between a transponder and the receivers, from which the relative position can be recovered [14]. The inertial measurements can be provided, e.g., by a LBL, or by an USBL positioning system. Let $\{B_i\}$ denote a coordinate frame attached to AUV i, denominated in the sequel as the body-fixed coordinate frame associated with the i-th AUV. The linear motion of AUV i can be written as

$$\dot{\mathbf{p}}_i(t) = \mathbf{R}_i(t)\mathbf{v}_i(t), \tag{13}$$

where $\mathbf{p}_i(t) \in \mathbb{R}^3$ is the inertial position of the vehicle, $\mathbf{v}_i(t) \in \mathbb{R}^3$ denotes its velocity relative to $\{I\}$, expressed in body-fixed coordinates of the i-th AUV, and $\mathbf{R}_i(t) \in SO(3)$ is the rotation matrix from $\{B_i\}$ to $\{I\}$, which satisfies $\dot{\mathbf{R}}_i(t) = \mathbf{R}_i(t)\mathbf{S}(\boldsymbol{\omega}_i(t))$, where $\boldsymbol{\omega}_i(t) \in \mathbb{R}^3$ is the angular velocity of $\{B_i\}$, expressed in body-fixed coordinates of the i-th AUV, and $\mathbf{S}(\boldsymbol{\omega})$ is the skew-symmetric matrix such that $\mathbf{S}(\boldsymbol{\omega})\boldsymbol{x}$ is the cross product $\boldsymbol{\omega} \times \boldsymbol{x}$. It is assumed that an Attitude and Heading Reference System (AHRS) installed on-board each AUV provides measurements of both $\mathbf{R}_i(t)$ and $\boldsymbol{\omega}_i(t)$. Additionally, suppose that each AUV has access to a linear acceleration measurement $\mathbf{a}_i(t) \in \mathbb{R}^3$, which follows

$$\mathbf{a}_i(t) = \dot{\mathbf{v}}_i(t) + \mathbf{S}(\boldsymbol{\omega}_i(t))\mathbf{v}_i(t) - \mathbf{g}_i(t), \tag{14}$$

where $\mathbf{g}_i(t) \in \mathbb{R}^3$ is the acceleration of gravity, expressed in body-fixed coordinates of the i-th AUV. Even though $\mathbf{g}_i(t)$ is usually well-known, it is treated as an unknown variable with practical applications in mind, where small errors in the estimation of the attitude of the vehicle may lead to significant errors in the acceleration compensation, see [4] for further details. Its time derivative is given by

$$\dot{\mathbf{g}}_i(t) = -\mathbf{S}(\boldsymbol{\omega}_i(t))\mathbf{g}_i(t). \tag{15}$$

For the first case, i.e., with inertial position readings, grouping equations (13), (14), and (15), and measuring the inertial position, yields the system

$$\begin{cases} \dot{\mathbf{p}}_i(t) = \mathbf{R}_i(t)\mathbf{v}_i(t) \\ \dot{\mathbf{v}}_i(t) = -\mathbf{S}(\boldsymbol{\omega}_i(t))\mathbf{v}_i(t) + \mathbf{g}_i(t) + \mathbf{a}_i(t) \\ \dot{\mathbf{g}}_i(t) = -\mathbf{S}(\boldsymbol{\omega}_i(t))\mathbf{g}_i(t) \\ \mathbf{y}_i(t) = \mathbf{p}_i(t) \end{cases}.$$

Using in each vehicle the Lyapunov state transformation introduced in [3],

$$\begin{bmatrix}\mathbf{x}_i^1(t)\\ \mathbf{x}_i^2(t)\\ \mathbf{x}_i^3(t)\end{bmatrix} := \mathbf{T}_i(t)\begin{bmatrix}\mathbf{p}_i(t)\\ \mathbf{v}_i(t)\\ \mathbf{g}_i(t)\end{bmatrix}, \tag{16}$$

with

$$\mathbf{T}_i(t) = \begin{bmatrix}\mathbf{I} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{R}_i(t) & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{R}_i(t)\end{bmatrix},$$

which preserves stability and observability properties [7], and making $\mathbf{u}_i(t) := \mathbf{R}_i(t)\mathbf{a}_i(t)$, the system dynamics can be written as the LTI system (1), with $n_L = 9$, $m_L = 3$, $o_L = 3$,

$$\mathbf{A}_L = \begin{bmatrix}\mathbf{0} & \mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{I}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix} \in \mathbb{R}^{n_L \times n_L},\ \mathbf{B}_L = \begin{bmatrix}\mathbf{0}\\ \mathbf{I}\\ \mathbf{0}\end{bmatrix} \in \mathbb{R}^{n_L \times m_L},$$

and $\mathbf{C}_L = \begin{bmatrix}\mathbf{I} & \mathbf{0} & \mathbf{0}\end{bmatrix} \in \mathbb{R}^{o_L \times n_L}$. Regarding the second case, i.e., when the AUV has access to relative position measurements and receives position estimates from the corresponding vehicles, a similar procedure can be carried out, yielding the system (3) instead, where $\mathbf{A}_L$ and $\mathbf{B}_L$ are defined as in the previous case, and $\mathbf{C}_i = \mathbf{I}_{N_i} \otimes \mathbf{C}_L$.

A. Filter Design

The use of noisy attitude and angular velocity measurements will inject multiplicative noise into the computations, so it is advantageous to implement the local observers in the body-fixed frame of their respective AUV to minimize its effect. To do so, consider for the first case (absolute measurements) the following continuous-discrete representation: denoting the state estimate by $\hat{\mathbf{x}}_i(t) \in \mathbb{R}^{n_L}$, its update between samples of the output follows

$$\dot{\hat{\mathbf{x}}}_i(t) = \mathbf{A}_L\hat{\mathbf{x}}_i(t) + \mathbf{B}_L\mathbf{u}_i(t).$$

Then, when the output sample k is obtained at time t_k, update the state following

$$\hat{\mathbf{x}}_i(t_k^+) = \hat{\mathbf{x}}_i(t_k) + \mathbf{L}_i(\mathbf{y}_i(t_k) - \mathbf{C}_L\hat{\mathbf{x}}_i(t_k)).$$

Finally, reverse the Lyapunov state transformation (16) by defining new state estimates $\hat{\mathbf{z}}_i(t) = \mathbf{T}_i^T(t)\hat{\mathbf{x}}_i(t)$. The update between samples of the output becomes

$$\dot{\hat{\mathbf{z}}}_i(t) = \mathbf{A}_i(t)\hat{\mathbf{z}}_i(t) + \mathbf{B}_L\mathbf{a}_i(t), \tag{17}$$

with

$$\mathbf{A}_i(t) = \begin{bmatrix}\mathbf{0} & \mathbf{R}_i(t) & \mathbf{0}\\ \mathbf{0} & -\mathbf{S}(\boldsymbol{\omega}_i(t)) & \mathbf{I}\\ \mathbf{0} & \mathbf{0} & -\mathbf{S}(\boldsymbol{\omega}_i(t))\end{bmatrix} \in \mathbb{R}^{n \times n}.$$

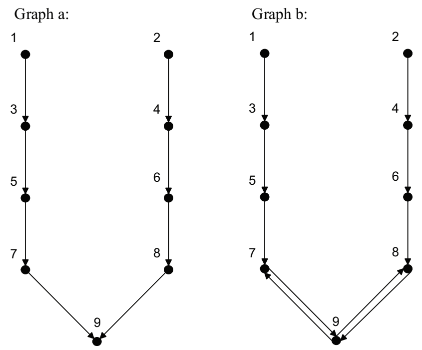

Fig. 1. Two formation graphs used in the simulations

When an output sample is available, update following

$$\hat{\mathbf{z}}_i(t_k^+) = \hat{\mathbf{z}}_i(t_k) + \mathbf{T}_i^T(t_k)\mathbf{L}_i(\mathbf{y}_i(t_k) - \mathbf{C}_L\hat{\mathbf{z}}_i(t_k)).$$

The same thing can be done for the AUVs with relative measurements: between output samples, update the state estimates following (17), and when position measurements are obtained, update following

$$\hat{\mathbf{z}}_i(t_k^+) = \hat{\mathbf{z}}_i(t_k) + \mathbf{T}_i^T(t_k)\mathbf{L}_i(\mathbf{y}_i(t_k) - \mathbf{C}_i\Delta\hat{\mathbf{z}}_i(t_k)).$$

This allows for filter implementation in the body-fixed coordinate frame of each AUV and takes into account the different sampling rates between on-board instrumentation (accelerometer, AHRS) and the positioning system.

V. Simulation Results

In the simulations that were carried out, two similar formation structures were considered, with associated graphs depicted in Fig. 1. Note that, while graph (a) is acyclic, (b) has two additional edges which create cycles in the graph. All sensor data was corrupted by additive, uncorrelated, zero-mean white Gaussian noise, with appropriate standard deviations given the equipment that would usually provide those measurements. Regarding the positioning data, the standard deviation was set to $1\ (m)$ for the relative measurements, and $0.1\ (m)$ for the absolute measurements. Furthermore, some cross-correlation was added to the noise on the absolute measurements to account for possible similarities in the source of the data (such as both AUVs using GPS at the surface, or using a LBL positioning system and sharing the same set of landmarks), resulting in the covariance matrix:

$$\mathbf{\Theta}_0 = 0.01 \times \begin{bmatrix} 1 & 0.1 \\ 0.1 & 1 \end{bmatrix} \otimes \mathbf{I}_3.$$

For the other measurements, the standard deviation of the white noise was set as follows:

- Linear acceleration - $0.01\ (m/s^2)$;
- Angular velocity - $0.05\ (°/s)$;
- Attitude, parametrized by Euler angles - $0.03\ (°)$ for the roll and pitch, $0.3\ (°)$ for the yaw.

A. H_2 norm minimization

To tune the optimization algorithm $\mathbf{\Xi}$ and $\mathbf{\Theta}$ were set to

$$\begin{cases} \mathbf{\Xi} = \text{diag}\left(\mathbf{\Xi}_L, \quad \mathbf{\Xi}_L, \quad \ldots, \quad \mathbf{\Xi}_L\right) \\ \mathbf{\Theta} = \text{diag}\left(\mathbf{\Theta}_0, \quad \mathbf{I}, \quad \mathbf{I}, \quad \ldots, \quad \mathbf{I}\right) \end{cases},$$

where $\mathbf{\Xi}_L = \text{diag}\left(0.001 \times \mathbf{I}_3, \quad 0.0001 \times \mathbf{I}_6\right)$. The algorithm was executed for both formation graphs, using in each several different sets of initial values for $\mathbf{L}$ computed through application of Theorem 1. The evolution of the objective function (μ) is depicted in Fig. 2 for the acyclic graph, and in Fig. 3 for the cyclic graph. In both images, the straight red line depicts the value achieved by the corresponding centralized Kalman filter. Table II details the value computed for μ for both graphs after 1000 iterations of the algorithm, as well as the value achieved by their centralized counterparts. The results show that the algorithm improved the performance of the decentralized estimator very significantly in both cases, and for all the initial conditions that were tested. Furthermore, the lower value achieved for the cyclic graph suggests that the additional edges were incorporated constructively by the algorithm (the corresponding gains were set to zero in the initial $\mathbf{L}$). In both cases, the achieved performance index is significantly worse for the decentralized state estimator than for the centralized Kalman filter, which is to be expected given the vastly superior amount of information available to the latter.

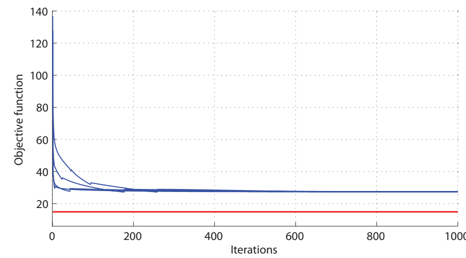

Fig. 2. Evolution of the algorithm for the acyclic graph

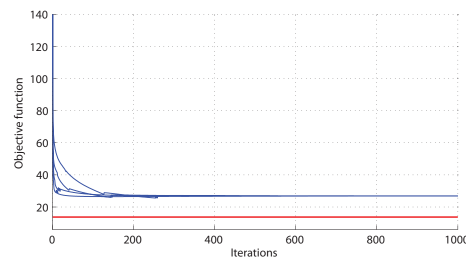

Fig. 3. Evolution of the algorithm for the cyclic graph

B. Performance assessment and comparison

The simulations were carried out for four different state estimators:

1) A decentralized state estimator based on the acyclic formation graph (Fig. 1, graph a), with the best gains found through the application of the proposed H_2 norm minimization algorithm.
2) A decentralized state estimator based on the cyclic formation graph (Fig. 1, graph b), with the best gains found through the application of the proposed H_2 norm minimization algorithm.

TABLE II

Lowest value achieved for μ

	Acy./Decent.	Cyc./Decent.	Acy./Cent.	Cyc./Cent.
μ_{min}	27.42	26.88	14.93	13.72

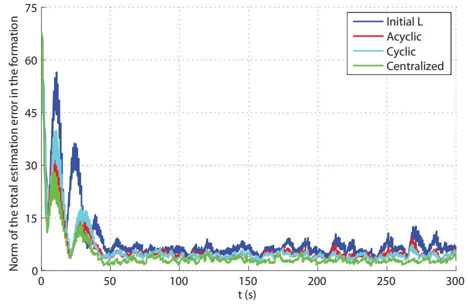

Fig. 4. Evolution of the norm of the total estimation error in the formation

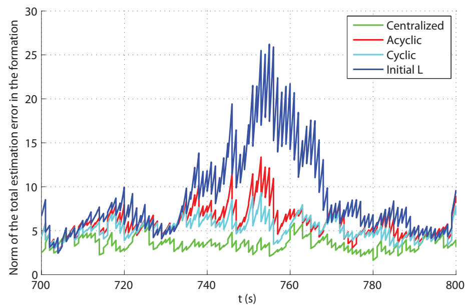

Fig. 5. Detailed view of the norm of the total estimation error in the formation, once the initial transients have vanished

3) A centralized Kalman filter based on the cyclic formation graph, to provide a lower bound for the attainable performance.
4) A decentralized estimator using gains obtained by straightforward application of Theorem 1. In this case, one of the gains computed to provide initial values for the H_2 minimization algorithm was used.

The evolution of the norm of the vector composed by all estimation error variables in the formation is depicted in Fig. 4, while its steady-state behavior is detailed in Fig. 5. The performance of the non-optimized filter is markedly worse than that of the other solutions, while the centralized Kalman filter achieves, as expected, the best performance. The signals depicted in Fig. 5 exhibit a characteristic sawtooth shape which results from the continuous-discrete filter implementation. Between samples of the output the state estimate is continuously updated in open loop, which naturally increases the error. When an output sample is obtained, it is used to correct the state estimate resulting in a sudden, sharp decay of the estimation error. Closer inspection of the graph of Fig. 5 also shows that the overall performance is better for the decentralized state observer based on the cyclic graph than for the one based on the acyclic graph, confirming that the additional edges present in the cyclic graph were incorporated constructively by the algorithm.

VI. Conclusions

This paper addressed the problem of state estimation in formations of autonomous vehicles. The approach considered here consists in the implementation of a local state observer in each vehicle relying only on locally available measurements and data communicated by neighboring agents, resulting in a decentralized state observer which features lower computational and communication loads than comparable centralized solutions. A method for computing observer gains which yield globally asymptotically stable error dynamics was presented for fixed topology formations, as well as an iterative algorithm for improving the decentralized estimator's performance when the measurements are corrupted by noise. The proposed framework was particularized to the practical case of a formation of Autonomous Underwater Vehicles (AUVs), and a continuous-discrete formulation was achieved for the local state observers, to take into account the difference in sampling rates between on-board instrumentation and positioning systems. To assess the performance of the solution, simulation results were presented and discussed for different formation topologies.

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