2013 European Control Conference (ECC)
July 17-19, 2013, Zürich, Switzerland.

GES Tightly Coupled Attitude Estimation based on a LBL/USBL Positioning System

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Abstract— **Typical attitude estimation solutions for underwater vehicles rely on magnetometers, which are prone to magnetic field distortions. This can preclude its use in intervention scenarios, in the vicinity of objects with strong magnetic signatures, severely endangering not only the intervention mission but also the operation of the underwater vehicle. This paper presents a novel attitude estimation solution, based on a combined Long Baseline / Ultra Short Baseline (LBL/USBL) acoustic positioning system, with application to underwater vehicles. The range and range differences of arrival obtained with the LBL/USBL are directly embedded in the estimator dynamics, without any linearization whatsoever, and globally exponentially stable (GES) error dynamics are achieved. Simulation results evidence good performance of the proposed solution.**

I. Introduction

The topic of attitude estimation is still very active, as evidenced by numerous recent publications, see e.g. [1], [2], [3], [4]. The Extended Kalman Filter (EKF) has been instrumental to many stochastic solutions, see e.g. [5], while nonlinear alternatives, aiming for stability and convergence properties, in deterministic settings, have been proposed in [6], [7], [8], [9], [10], [11], and [12], to mention just a few, see [13] for a thorough survey on attitude estimation. Recently, the authors have proposed two alternative solutions in [14] and [15]. In the first, the Kalman filter is the workhorse, where no linearizations are carried out whatsoever, resulting in a design which guarantees globally asymptotically stable (GAS) error dynamics. In the later, a cascade observer is proposed that achieves globally exponentially stable (GES) error dynamics and that requires less computational power than the Kalman filter, at the expense of the filtering performance. Common to both solutions is the fact that the topological restrictions of the Special Orthogonal Group $SO(3)$ are not explicitly imposed, though they are verified asymptotically in the absence of noise. In the presence of sensor noise, the distance of the estimates provided by the cascade observer or the Kalman filter to $SO(3)$ remains close to zero and methods are proposed that give estimates of the attitude arbitrarily close to $SO(3)$. In [16] an alternative additional result gives attitude estimates explicitly on $SO(3)$, at the possible expense of continuity of the solution during the initial transients, hence not violating the topological limitations that are thoroughly discussed in [17].

This work was partially supported by the FCT [PEst-OE/EEI/LA0009/2011] and by the EU Project TRIDENT (Contract No. 248497).

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For underwater vehicles, the usual sensing devices employed for attitude determination are two triads of orthogonally mounted accelerometers and magnetometers, coupled with a triad of orthogonally mounted rate gyros, employed for filtering purposes. Essentially, the magnetometers and the accelerometers provide direct measurements, in body-fixed coordinates, of known vectors in inertial coordinates. Hence, an attitude estimate can be readily obtained from the solution of the Wahba's problem. With additional angular velocity measurements, it is then possible to design an attitude filter, possibly including the estimation of rate gyro bias. The disadvantage of the use of magnetometers is that they are subject to magnetic field anomalies, such as the ones that can be encountered nearby objects with strong magnetic signatures, rendering the magnetic field measurements useless. This can be particularly dangerous in underwater intervention scenarios and as such alternatives need to be devised.

In previous work by the authors, see [18], a novel complete navigation system was proposed based on a combined Long Baseline / Ultra-short Baseline (LBL/USBL) acoustic positing system. In short, with a LBL/USBL it is not only possible to determine the inertial position of the vehicle but also the positions of the external LBL landmarks with respect to the vehicle, expressed in body-fixed coordinates. In [18], and for attitude estimation purposes, the later were employed to obtain body-fixed vector measurements of known constant inertial vectors, hence allowing for attitude estimation.

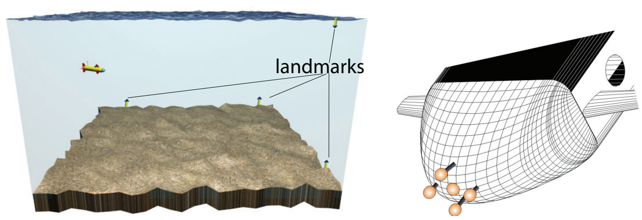
The actual measurements of a LBL/USBL acoustic positioning system are acoustic signals, which when processed yield ranges and range differences of arrival between the acoustic receivers of the USBL. With some computations, it is possible to obtain the measurements required for the attitude estimation solution proposed in [18]. However, it would be beneficial if the actual range and range differences of arrival could be directly employed in the attitude estimation solution, avoiding intermediate nonlinear computations that can distort noise and allowing for better tuning of the estimator parameters. The main contribution of this paper is the design of a tightly coupled attitude estimation solution based on a LBL/USBL acoustic positioning system. The range and range differences of arrival are used directly in the observer feedback loop, hence avoiding intermediate computations, and no linearizations are carried out whatsoever. The proposed observer achieves globally exponentially stable error dynamics and it is computationally efficient. Topological limitations are avoided by relaxation of the constraints of the Special Orthogonal Group, which are nevertheless verified asymptotically.

A. Notation

The symbol $\mathbf{0}$ denotes a matrix (or vector) of zeros, $\mathbf{I}$ the identity matrix, and $\mathbf{blkdiag}(\mathbf{A}_1, \ldots, \mathbf{A}_n)$ a block diagonal matrix, all assumed of appropriate dimensions. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, the cross and inner products are represented by $\mathbf{x} \times \mathbf{y}$ and $\mathbf{x} \cdot \mathbf{y}$, respectively.

II. Problem statement

Consider an underwater vehicle moving in a scenario where there is a set of fixed landmarks installed in a Long Baseline configuration and suppose that the vehicle is equipped with an Ultra Short Baseline acoustic positioning system, which measures not only the distance between the vehicle and each landmark but also the range differences of arrival between the acoustic receivers of the USBL, from each landmark, as depicted in Fig. 1. For further details on the USBL, the reader is referred to [19], [20], and references therein. Further assume that the vehicle is equipped with a triad of orthogonally mounted rate gyros. The problem considered in this paper is the design of a highly integrated sensor-based solution to estimate the attitude of the vehicle and the rate gyro bias.



(a) AUV and LBL array (b) AUV with USBL array

Fig. 1. Mission Scenario

A. System dynamics

In order to set the problem framework, let $\{I\}$ denote a local inertial reference coordinate frame and $\{B\}$ a coordinate frame attached to the vehicle, commonly denominated as the body-fixed reference frame. The linear motion of the vehicle is described by $\dot{\mathbf{p}}(t) = \mathbf{R}(t)\mathbf{v}(t)$, where $\mathbf{p}(t) \in \mathbb{R}^3$ denotes the inertial position of the vehicle, $\mathbf{v}(t) \in \mathbb{R}^3$ is the velocity of the vehicle relative to $\{I\}$ and expressed in body-fixed coordinates, and $\mathbf{R}(t) \in SO(3)$ is the rotation matrix from $\{B\}$ to $\{I\}$, which satisfies

$$\dot{\mathbf{R}}(t) = \mathbf{R}(t)\mathbf{S}\left(\boldsymbol{\omega}(t)\right), \tag{1}$$

where $\boldsymbol{\omega}(t) \in \mathbb{R}^3$ is the angular velocity of $\{B\}$, expressed in body-fixed coordinates, and $\mathbf{S}(\boldsymbol{\omega})$ is the skew-symmetric matrix such that $\mathbf{S}(\boldsymbol{\omega})\boldsymbol{x}$ is the cross product $\boldsymbol{\omega} \times \boldsymbol{x}$.

Let $\mathbf{s}_i \in \mathbb{R}^3$, $i = 1, \ldots, N$, denote the inertial positions of known landmarks, and $\mathbf{a}_i \in \mathbb{R}^3$, $i = 1, \ldots, M$, the positions of the array of receivers of the USBL relative to the origin of $\{B\}$, expressed in body-fixed coordinates. Then, the range measurement between the i-th landmark and the j-th acoustic receiver of the USBL is given by

$$r_{i,j}(t) = \|\mathbf{s}_i - \mathbf{p}(t) - \mathbf{R}(t)\mathbf{a}_j\| \in \mathbb{R}. \tag{2}$$

The rate gyro measurements $\boldsymbol{\omega}_m(t)$ satisfy

$$\boldsymbol{\omega}_m(t) = \boldsymbol{\omega}(t) + \mathbf{b}_{\boldsymbol{\omega}}(t), \tag{3}$$

where $\mathbf{b}_{\boldsymbol{\omega}}(t) \in \mathbb{R}^3$ denotes the rate gyro bias, which is assumed constant, i.e.,

$$\dot{\mathbf{b}}_{\boldsymbol{\omega}}(t) = \mathbf{0}. \tag{4}$$

Using (3) and combining (1), (2), and (4) yields

$$\begin{cases} \dot{\mathbf{R}}(t) = \mathbf{R}(t)\mathbf{S}\left(\boldsymbol{\omega}_m(t) - \mathbf{b}_{\boldsymbol{\omega}}(t)\right) \\ \dot{\mathbf{b}}_{\boldsymbol{\omega}}(t) = \mathbf{0} \\ r_{1,1}(t) = \|\mathbf{s}_1 - \mathbf{p}(t) - \mathbf{R}(t)\mathbf{b}_1\| \\ \vdots \\ r_{N,M}(t) = \|\mathbf{s}_N - \mathbf{p}(t) - \mathbf{R}(t)\mathbf{b}_M\| \end{cases}. \tag{5}$$

The problem considered in the paper is the design of an observer for (5).

B. Long Baseline / Ultra Short Baseline configuration

Throughout the paper the following standard assumptions are considered.

Assumption 1: The LBL acoustic positioning system includes at least 4 noncoplanar landmarks and the distance between the landmarks of the LBL is much larger than the distance between the receivers of the USBL acoustic positioning system.

Assumption 2: The USBL acoustic positioning system includes at least 4 noncoplanar receivers and the distance between the landmarks of the LBL is much larger than the distance between the receivers of the USBL acoustic positioning system.

III. Observer design

This section details the design of an attitude observer that uses directly the ranges and range differences of arrival and that achieves globally exponentially stable error dynamics. The proposed approach builds vaguely on two different methodologies previously proposed by the authors. First, a sensor-based observer for the rate gyro bias is developed by appropriate state definition, which bears some resemblance with the design proposed in [21], where the problems of source localization and navigation based on single range measurements were addressed. Secondly, a cascade attitude observer is proposed assuming that the rate gyro bias is known. Finally, the overall cascade observer is proposed and its stability is analyzed. The cascade design is similar, at large, to that proposed in [15]. However, the structures of each individual observer are very different as they now rely on range and range differences of arrival measurements instead of vector measurements.

A. Rate gyro bias observer

The dependence of the attitude observer (and, consequently, the bias observer) on the inertial position of the vehicle is highly undesirable and in fact it should not be required. Indeed, in a LBL/USBL framework, the positions of the LBL landmarks with respect to the vehicle, expressed in body-fixed coordinates, are indirectly available (after some computations). If one takes the difference between pairs of these vectors, one obtains a set of body-fixed vectors that correspond to constant known inertial vectors, obtained from the differences of the inertial positions of the LBL landmarks. As such, this information suffices to determine the attitude of the vehicle without the need of the inertial position of the vehicle. In fact, this is the idea of the approach proposed in [18]. This paper aims at achieving the same result but using directly the ranges and range differences of arrival.

Let $\mathcal{C}_{\mathbf{s}}$ denote a set of 2-combinations of elements of the set $\{1,\ldots,N\}$, e.g.
$\mathcal{C}_{\mathbf{s}}=\{(1,2),\ldots(1,N),(2,3),\ldots,(2,N),\ldots,(N-1,N)\}$, and let $\mathcal{C}_{\mathbf{a}}$ denote a set of 2-combinations of elements of the set $\{1,\ldots,M\}$, e.g.
$\mathcal{C}_{\mathbf{a}}=\{(1,2),\ldots(1,M),(2,3),\ldots,(2,M),\ldots,(M-1,M)\}$.
Define

$$q(m,n,i,j,t) \;:=\; -\frac{1}{2}\left[r_{m,i}^2(t)-r_{n,i}^2(t)\right] +\frac{1}{2}\left[r_{m,j}^2(t)-r_{n,j}^2(t)\right] \tag{6}$$

for all $(m,n,i,j)\in\mathcal{C}_{\mathbf{s}}\times\mathcal{C}_{\mathbf{a}}$. First, notice that $q(m,n,i,j,t)$ is a direct function of the ranges and range differences of arrival, as it is possible to rewrite it as

$$q(m,n,i,j,t)=\tfrac{1}{2}\left[r_{n,i}(t)+r_{n,j}(t)\right]\left[r_{n,i}(t)-r_{n,j}(t)\right] -\tfrac{1}{2}\left[r_{m,i}(t)+r_{m,j}(t)\right]\left[r_{m,i}(t)-r_{m,j}(t)\right].$$

Next, substituting (2) in (6) gives

$$q(m,n,i,j,t)=\left(\mathbf{s}_m-\mathbf{s}_n\right)^T\mathbf{R}(t)\left(\mathbf{a}_i-\mathbf{a}_j\right). \tag{7}$$

As it can be seen, the inertial position of the vehicle does not influence $q(m,n,i,j,t)$. Yet, it depends on the attitude of the vehicle and, considering all 2-combinations of LBL landmarks and all 2-combinations of USBL receivers, it is related to the entire geometric structure of the LBL/USBL positioning system. The idea of the bias observer is to use $q(m,n,i,j,t)$, for all $(m,n,i,j)\in\mathcal{C}_{\mathbf{s}}\times\mathcal{C}_{\mathbf{a}}$, as system states, which are measured, in order to estimate the rate gyro bias $\mathbf{b}_{\boldsymbol{\omega}}(t)$, which is unknown.

Before proceeding some additional definitions are required. In particular, define, for all $(i,j)\in\mathcal{C}_{\mathbf{a}}$, additional unit vectors $\mathbf{a}_{i,j}^{\perp_1}\in\mathbb{R}^3$ and $\mathbf{a}_{i,j}^{\perp_2}\in\mathbb{R}^3$ such that

$$\left\{\begin{array}{l} \frac{\mathbf{a}_i-\mathbf{a}_j}{\|\mathbf{a}_i-\mathbf{a}_j\|}\times\mathbf{a}_{i,j}^{\perp_1}=\mathbf{a}_{i,j}^{\perp_2}\\ \mathbf{a}_{i,j}^{\perp_1}\times\mathbf{a}_{i,j}^{\perp_2}=\frac{\mathbf{a}_i-\mathbf{a}_j}{\|\mathbf{a}_i-\mathbf{a}_j\|}\\ \mathbf{a}_{i,j}^{\perp_2}\times\frac{\mathbf{a}_i-\mathbf{a}_j}{\|\mathbf{a}_i-\mathbf{a}_j\|}=\mathbf{a}_{i,j}^{\perp_1} \end{array}\right. . \tag{8}$$

In short, the sets of vectors $\left\{\frac{\mathbf{a}_i-\mathbf{a}_j}{\|\mathbf{a}_i-\mathbf{a}_j\|},\mathbf{a}_{i,j}^{\perp_1},\mathbf{a}_{i,j}^{\perp_2}\right\}$, for all $(i,j)\in\mathcal{C}_{\mathbf{a}}$, form orthonormal bases of $\mathbf{R}^3$. Next, notice that under Assumption 2, it is always possible to express all additional vectors $\mathbf{a}_{i,j}^{\perp_1}$ and $\mathbf{a}_{i,j}^{\perp_2}$ as a linear combination of vectors $\mathbf{a}_k-\mathbf{a}_l$. Let these be defined as

$$\left\{\begin{array}{l} \mathbf{a}_{i,j}^{\perp_1}=\sum\limits_{(k,l)\in\mathcal{C}_{\mathbf{a}}}\phi_1(i,j,k,l)\left(\mathbf{a}_k-\mathbf{a}_l\right)\\ \mathbf{a}_{i,j}^{\perp_2}=\sum\limits_{(k,l)\in\mathcal{C}_{\mathbf{a}}}\phi_2(i,j,k,l)\left(\mathbf{a}_k-\mathbf{a}_l\right) \end{array}\right. \tag{9}$$

for all $(i,j)\in\mathcal{C}_{\mathbf{a}}$, where $\phi_1(i,j,k,l),\phi_2(i,j,k,l)\in\mathbb{R}$ are the linear combination coefficients.

The nominal system dynamics of the rate gyro bias observer are now derived. Taking the derivative of (7), and using (5), gives

$$\dot{q}(m,n,i,j,t)=\left(\mathbf{s}_m-\mathbf{s}_n\right)^T\mathbf{R}(t)\mathbf{S}\left(\boldsymbol{\omega}_m(t)\right)\left(\mathbf{a}_i-\mathbf{a}_j\right) -\left(\mathbf{s}_m-\mathbf{s}_n\right)^T\mathbf{R}(t)\mathbf{S}\left(\mathbf{b}_{\boldsymbol{\omega}}(t)\right)\left(\mathbf{a}_i-\mathbf{a}_j\right). \tag{10}$$

Express $\boldsymbol{\omega}_m(t)$ as the linear combination

$$\boldsymbol{\omega}_m(t) \;=\; \boldsymbol{\omega}_m(t)\cdot\frac{\left(\mathbf{a}_i-\mathbf{a}_j\right)}{\|\left(\mathbf{a}_i-\mathbf{a}_j\right)\|}\frac{\left(\mathbf{a}_i-\mathbf{a}_j\right)}{\|\left(\mathbf{a}_i-\mathbf{a}_j\right)\|} +\boldsymbol{\omega}_m(t)\cdot\mathbf{a}_{i,j}^{\perp_1}\mathbf{a}_{i,j}^{\perp_1}+\boldsymbol{\omega}_m(t)\cdot\mathbf{a}_{i,j}^{\perp_2}\mathbf{a}_{i,j}^{\perp_2}. \tag{11}$$

Using (11) first and then (8) it is possible to write

$$\boldsymbol{\omega}_m(t)\times\left(\mathbf{a}_i-\mathbf{a}_j\right)=\boldsymbol{\omega}_m(t)\cdot\mathbf{a}_{i,j}^{\perp_2}\|\mathbf{a}_i-\mathbf{a}_j\|\mathbf{a}_{i,j}^{\perp_1} -\boldsymbol{\omega}_m(t)\cdot\mathbf{a}_{i,j}^{\perp_1}\|\mathbf{a}_i-\mathbf{a}_j\|\mathbf{a}_{i,j}^{\perp_2}. \tag{12}$$

Substituting (9) in (12) gives

$$\begin{aligned}\boldsymbol{\omega}_m(t)\times\left(\mathbf{a}_i-\mathbf{a}_j\right)=&\\ \boldsymbol{\omega}_m(t)\cdot\mathbf{a}_{i,j}^{\perp_2}\|\mathbf{a}_i-\mathbf{a}_j\|&\sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}}\phi_1(i,j,k,l)\left(\mathbf{a}_k-\mathbf{a}_l\right)\\ -\boldsymbol{\omega}_m(t)\cdot\mathbf{a}_{i,j}^{\perp_1}\|\mathbf{a}_i-\mathbf{a}_j\|&\sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}}\phi_2(i,j,k,l)\left(\mathbf{a}_k-\mathbf{a}_l\right).\end{aligned} \tag{13}$$

Substituting (13) in the first term of the right side of (10), using (7), and following the same circle of ideas for the second term gives the nonlinear dynamics

$$\begin{aligned}&\dot{q}(m,n,i,j,t)=\\ &\boldsymbol{\omega}_m(t)\cdot\mathbf{a}_{i,j}^{\perp_2}\|\mathbf{a}_i-\mathbf{a}_j\|\sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}}\phi_1(i,j,k,l)\,q(m,n,k,l,t)\\ &-\boldsymbol{\omega}_m(t)\cdot\mathbf{a}_{i,j}^{\perp_1}\|\mathbf{a}_i-\mathbf{a}_j\|\sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}}\phi_2(i,j,k,l)\,q(m,n,k,l,t)\\ &+\mathbf{b}_{\boldsymbol{\omega}}(t)\cdot\mathbf{a}_{i,j}^{\perp_1}\|\mathbf{a}_i-\mathbf{a}_j\|\sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}}\phi_2(i,j,k,l)\,q(m,n,k,l,t)\\ &-\mathbf{b}_{\boldsymbol{\omega}}(t)\cdot\mathbf{a}_{i,j}^{\perp_2}\|\mathbf{a}_i-\mathbf{a}_j\|\sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}}\phi_1(i,j,k,l)\,q(m,n,k,l,t).\end{aligned} \tag{14}$$

for all $(m,n,i,j)\in\mathcal{C}_{\mathbf{s}}\times\mathcal{C}_{\mathbf{a}}$. Notice that (14) depends only on the USBL array geometry, the rate gyro measurements $\boldsymbol{\omega}_m(t)$, the additional quantities $q(m,n,i,j,t)$, the linear coefficients $\phi_1(i,j,k,l)$ and $\phi_2(i,j,k,l)$, all available, and the unknown rate gyro bias $\mathbf{b}_{\boldsymbol{\omega}}(t)$.

Consider the rate gyro bias observer dynamics given by

$$\begin{aligned}&\dot{\hat{q}}(m,n,i,j,t)=\\ &\boldsymbol{\omega}_m(t)\cdot\mathbf{a}_{i,j}^{\perp_2}\|\mathbf{a}_i-\mathbf{a}_j\|\sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}}\phi_1(i,j,k,l)\,\hat{q}(m,n,k,l,t)\\ &-\boldsymbol{\omega}_m(t)\cdot\mathbf{a}_{i,j}^{\perp_1}\|\mathbf{a}_i-\mathbf{a}_j\|\sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}}\phi_2(i,j,k,l)\,\hat{q}(m,n,k,l,t)\\ &+\hat{\mathbf{b}}_{\boldsymbol{\omega}}(t)\cdot\mathbf{a}_{i,j}^{\perp_1}\|\mathbf{a}_i-\mathbf{a}_j\|\sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}}\phi_2(i,j,k,l)\,q(m,n,k,l,t)\\ &-\hat{\mathbf{b}}_{\boldsymbol{\omega}}(t)\cdot\mathbf{a}_{i,j}^{\perp_2}\|\mathbf{a}_i-\mathbf{a}_j\|\sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}}\phi_1(i,j,k,l)\,q(m,n,k,l,t)\\ &+\alpha(m,n,i,j)\left[q(m,n,i,j,t)-\hat{q}(m,n,i,j,t)\right]\end{aligned} \tag{15}$$

for all $(m,n,i,j)\in\mathcal{C}_{\mathbf{s}}\times\mathcal{C}_{\mathbf{a}}$, and

$$\begin{aligned}\dot{\hat{\mathbf{b}}}_{\boldsymbol{\omega}}(t)=&\sum_{(m,n,i,j)\in\mathcal{C}_{\mathbf{s}}\times\mathcal{C}_{\mathbf{a}}}\beta(m,n,i,j)\\ &\|\mathbf{a}_i-\mathbf{a}_j\|\left[q(m,n,i,j,t)-\hat{q}(m,n,i,j,t)\right]\\ &\left[\mathbf{a}_{i,j}^{\perp_1}\sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}}\phi_2(i,j,k,l)\,q(m,n,k,l,t)\right.\\ &\left.-\mathbf{a}_{i,j}^{\perp_2}\sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}}\phi_1(i,j,k,l)\,q(m,n,k,l,t)\right],\end{aligned} \tag{16}$$

where $\alpha(m,n,i,j)>0$ and $\beta(m,n,i,j)>0$, for all $(m,n,i,j)\in\mathcal{C}_{\mathbf{s}}\times\mathcal{C}_{\mathbf{a}}$, are observer tuning parameters.

Let $\tilde{q}(m,n,i,j,t):=q(m,n,i,j,t)-\hat{q}(m,n,i,j,t)$, for all $(m,n,i,j)\in\mathcal{C}_{\mathbf{s}}\times\mathcal{C}_{\mathbf{a}}$ and $\tilde{\mathbf{b}}_{\boldsymbol{\omega}}(t):=\mathbf{b}_{\boldsymbol{\omega}}(t)-\hat{\mathbf{b}}_{\boldsymbol{\omega}}(t)$ denote the observer error. Then, the observer error dynamics are given by

$$\dot{\tilde{q}}(m,n,i,j,t) =$$
$$\boldsymbol{\omega}_m(t)\cdot\mathbf{a}_{i,j}^{\perp_2}\|\mathbf{a}_i-\mathbf{a}_j\|\sum_{(k,l)\in\mathcal{C}_\mathbf{a}}\phi_1(i,j,k,l)\,\tilde{q}(m,n,k,l,t)$$
$$-\boldsymbol{\omega}_m(t)\cdot\mathbf{a}_{i,j}^{\perp_1}\|\mathbf{a}_i-\mathbf{a}_j\|\sum_{(k,l)\in\mathcal{C}_\mathbf{a}}\phi_2(i,j,k,l)\,\tilde{q}(m,n,k,l,t)$$
$$+\tilde{\mathbf{b}}_{\boldsymbol{\omega}}(t)\cdot\mathbf{a}_{i,j}^{\perp_1}\|\mathbf{a}_i-\mathbf{a}_j\|\sum_{(k,l)\in\mathcal{C}_\mathbf{a}}\phi_2(i,j,k,l)\,q(m,n,k,l,t)$$
$$-\tilde{\mathbf{b}}_{\boldsymbol{\omega}}(t)\cdot\mathbf{a}_{i,j}^{\perp_2}\|\mathbf{a}_i-\mathbf{a}_j\|\sum_{(k,l)\in\mathcal{C}_\mathbf{a}}\phi_1(i,j,k,l)\,q(m,n,k,l,t)$$
$$-\alpha(m,n,i,j)\,\tilde{q}(m,n,i,j,t)$$

for all $(m,n,i,j)\in\mathcal{C}_\mathbf{s}\times\mathcal{C}_\mathbf{a}$, and

$$\dot{\tilde{\mathbf{b}}}_{\boldsymbol{\omega}}(t) = -\sum_{(m,n,i,j)\in\mathcal{C}_\mathbf{s}\times\mathcal{C}_\mathbf{a}}\beta(m,n,i,j)\,\tilde{q}(m,n,i,j,t)$$
$$\|\mathbf{a}_i-\mathbf{a}_j\|\left[\mathbf{a}_{i,j}^{\perp_1}\sum_{(k,l)\in\mathcal{C}_\mathbf{a}}\phi_2(i,j,k,l)\,q(m,n,k,l,t)\right.$$
$$\left.-\mathbf{a}_{i,j}^{\perp_2}\sum_{(k,l)\in\mathcal{C}_\mathbf{a}}\phi_1(i,j,k,l)\,q(m,n,k,l,t)\right].$$

The following theorem establishes that the resulting rate gyro bias observer has globally exponentially stable error dynamics.

Theorem 1: Suppose that Assumptions 1 and 2 are satisfied and consider the rate gyro bias observer given by (15) and (16), where $\alpha(m,n,i,j)>0$ and $\beta(m,n,i,j)>0$ for all $(m,n,i,j)\in\mathcal{C}_\mathbf{s}\times\mathcal{C}_\mathbf{a}$. Then, the origin of the error dynamics is a globally exponentially stable equilibrium point.

Proof: Let

$$\tilde{\boldsymbol{x}}_1(t) := \begin{bmatrix}\vdots\\ \tilde{q}(m,n,i,j,t)\\ \vdots\\ \tilde{\mathbf{b}}_{\boldsymbol{\omega}}(t)\end{bmatrix}\in\mathbb{R}^{{}_2^NC\,{}_2^MC+3},$$

$(m,n,i,j)\in\mathcal{C}_\mathbf{s}\times\mathcal{C}_\mathbf{a}$, denote the estimator error, in compact form, where ${}_2^NC=N(N-1)/2$ and ${}_2^MC=M(M-1)/2$ denote the number of 2-combinations of N and M elements, respectively. Define

$$V_1(t) := \frac{1}{2}\sum_{(i,j,k,l)\in\mathcal{C}_\mathbf{s}\times\mathcal{C}_\mathbf{a}}\beta(m,n,i,j)\left[\tilde{q}(m,n,i,j,t)\right]^2 + \frac{1}{2}\left\|\tilde{\mathbf{b}}_{\boldsymbol{\omega}}(t)\right\|^2$$

as a Lyapunov function candidate. Clearly,

$$\gamma_1\|\tilde{\boldsymbol{x}}_1(t)\|^2\leq V_1(t)\leq\gamma_2\|\tilde{\boldsymbol{x}}_1(t)\|^2, \quad (17)$$

where $\gamma_1:=\frac{1}{2}\min(1,\beta(m,n,i,j)),(m,n,i,j)\in\mathcal{C}_\mathbf{s}\times\mathcal{C}_\mathbf{a}$ and $\gamma_2:=\frac{1}{2}\max(1,\beta(m,n,i,j)),(m,n,i,j)\in\mathcal{C}_\mathbf{s}\times\mathcal{C}_\mathbf{a}$. The time derivative of $V_1(t)$ can be written, after some straightforward computations, as

$$\dot{V}_1(t) = -\tilde{\boldsymbol{x}}_1^T(t)\mathbf{C}_1^T\mathbf{C}_1\tilde{\boldsymbol{x}}_1(t)$$
$$= -\sum_{(i,j,k,l)\in\mathcal{C}_\mathbf{s}\times\mathcal{C}_\mathbf{a}}\alpha(m,n,i,j)\,\beta(m,n,i,j)\left[\tilde{q}(m,n,i,j,t)\right]^2,$$

where $\mathbf{C}_1=\begin{bmatrix}\mathbf{blkdiag}\left(\sqrt{\alpha(\mathrm{m,n,i,j})\,\beta(\mathrm{m,n,i,j})}\right) & \mathbf{0}\end{bmatrix}$. Hence,

$$\dot{V}_1(t)\leq 0. \quad (18)$$

Now, notice that the error dynamics can be written as the linear time-varying (LTV) system

$$\dot{\tilde{\boldsymbol{x}}}_1(t) = \boldsymbol{\mathcal{A}}_1(t)\tilde{\boldsymbol{x}}_1(t), \quad (19)$$

where

$$\boldsymbol{\mathcal{A}}_1(t) = \begin{bmatrix}\boldsymbol{\mathcal{A}}_{11}(t) & \boldsymbol{\mathcal{A}}_{12}(t)\\ \boldsymbol{\mathcal{A}}_{21}(t) & \mathbf{0}\end{bmatrix}$$

and each row of the matrix $\boldsymbol{\mathcal{A}}_{12}(t)$, corresponding to the state error $\tilde{q}(m,n,i,j,t)$, is given by

$$\|\mathbf{a}_i-\mathbf{a}_j\|\sum_{(k,l)\in\mathcal{C}_\mathbf{a}}\phi_2(i,j,k,l)\,q(m,n,k,l,t)\left(\mathbf{a}_{i,j}^{\perp_1}\right)^T$$
$$-\|\mathbf{a}_i-\mathbf{a}_j\|\sum_{(k,l)\in\mathcal{C}_\mathbf{a}}\phi_1(i,j,k,l)\,q(m,n,k,l,t)\left(\mathbf{a}_{i,j}^{\perp_2}\right)^T.$$

The definitions of $\boldsymbol{\mathcal{A}}_{11}(t)$ and $\boldsymbol{\mathcal{A}}_{21}(t)$ are omitted as they are not required in the sequel. If in addition to (17) and (18), the pair $(\boldsymbol{\mathcal{A}}_1(t),\mathbf{C}_1)$ is uniformly completely observable, then the origin of the linear time-varying system (19) is a globally exponentially stable equilibrium point, see [22, Example 8.11]. The remainder of the proof amounts to show that the pair $(\boldsymbol{\mathcal{A}}_1(t),\mathbf{C}_1)$ is uniformly completely observable. For any piecewise continuous, bounded matrix $\mathbf{K}_1(t)$, of compatible dimensions, uniform complete observability of the pair $(\boldsymbol{\mathcal{A}}_1(t),\mathbf{C}_1)$ is equivalent to uniform complete observability of the pair $(\mathbf{A}_1(t),\mathbf{C}_1)$, with $\mathbf{A}_1(t):=\boldsymbol{\mathcal{A}}_1(t)-\mathbf{K}_1(t)\mathbf{C}_1$, see [23, Lemma 4.8.1]. Now, notice that, attending to the particular forms of $\mathbf{C}_1$ and $\boldsymbol{\mathcal{A}}_1(t)$, there exists a continuous bounded matrix $\mathbf{K}_1(t)$, which depends explicitly on the observer parameters, the rate gyro readings, $\boldsymbol{\omega}_m(t)$, the USBL structure, the linear coefficients $\phi_1(i,j,k,l)$ and $\phi_2(i,j,k,l)$, and $q(m,n,i,j,t)$, $(m,n,i,j)\in\mathcal{C}_\mathbf{s}\times\mathcal{C}_\mathbf{a}$, such that

$$\mathbf{A}_1(t) = \begin{bmatrix}\mathbf{0} & \boldsymbol{\mathcal{A}}_{12}(t)\\ \mathbf{0} & \mathbf{0}\end{bmatrix}.$$

The expression of $\mathbf{K}_1(t)$ is not presented here as it is evident from the context and it is not required in the sequel. It remains to show that the pair $(\mathbf{A}_1(t),\mathbf{C}_1)$ is uniformly completely observable, i.e., that there exist positive constants ϵ_1, ϵ_2, and δ such that

$$\epsilon_1\mathbf{I}\preceq\boldsymbol{\mathcal{W}}(t,t+\delta)\preceq\epsilon_2\mathbf{I} \quad (20)$$

for all $t\geq t_0$, where $\boldsymbol{\mathcal{W}}(t_0,t_f)$ is the observability Gramian associated with the pair $(\mathbf{A}_1(t),\mathbf{C}_1)$ on $[t_0,t_f]$. Since the entries of both $\mathbf{A}_1(t)$ and $\mathbf{C}_1$ are continuous and bounded, the right side of (20) is evidently verified. Therefore, only the left side of (20) requires verification. This is omitted due to space limitations and it will be included in an expanded version of the paper. ■

B. Attitude observer

Let $\mathbf{x}_2(t) := \begin{bmatrix}\mathbf{z}_1^T(t) & \mathbf{z}_2^T(t) & \mathbf{z}_3^T(t)\end{bmatrix}^T\in\mathbb{R}^9$ be a column representation of $\mathbf{R}(t)$, where

$$\mathbf{R}(t) = \begin{bmatrix}\mathbf{z}_1^T(t)\\ \mathbf{z}_2^T(t)\\ \mathbf{z}_3^T(t)\end{bmatrix},$$

with $\mathbf{z}_i(t)\in\mathbb{R}^3$, $i=1,2,3$. Then, it is easy to show that

$$\dot{\boldsymbol{x}}_2(t) = -\mathbf{S}_3(\boldsymbol{\omega}_m(t)-\mathbf{b}_{\boldsymbol{\omega}}(t))\,\boldsymbol{x}_2(t),$$

where $\mathbf{S}_3(\mathbf{x}):=\mathbf{blkdiag}(\mathbf{S}(\mathbf{x}),\mathbf{S}(\mathbf{x}),\mathbf{S}(\mathbf{x}))\in\mathbb{R}^{9\times 9}$.

From (7) it is possible to write $q\left(m,n,i,j,t\right)$ as a linear combination of elements of $\boldsymbol{x}_2(t)$, i.e.,

$$q\left(m,n,i,j,t\right) = \mathbf{c}_{m,n,i,j} \cdot \boldsymbol{x}_2(t),$$

where

$$\mathbf{c}_{m,n,i,j} := \begin{bmatrix} (\mathbf{a}_i - \mathbf{a}_j) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\mathbf{a}_i - \mathbf{a}_j) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (\mathbf{a}_i - \mathbf{a}_j) \end{bmatrix} (\mathbf{s}_m - \mathbf{s}_n) \in \mathbb{R}^9.$$

Let

$$\mathbf{q}(t) := \begin{bmatrix} \vdots \\ q\left(m,n,i,j,t\right) \\ \vdots \end{bmatrix} \in \mathbb{R}^{{}^M_2C\,{}^M_2C},$$

$(m,n,i,j) \in \mathcal{C}_\mathbf{s} \times \mathcal{C}_\mathbf{a}$. Then, it is possible to write

$$\mathbf{q}(t) = \mathbf{C}_2 \boldsymbol{x}_2(t),$$

where $\mathbf{C}_2 \in \mathbb{R}^{{}^N_2C\,{}^M_2C \times 9}$ is omitted as it is evident from the context. Under Assumptions 1 and 2 is is trivial to show that $\mathbf{C}_2$ has full rank.

Consider the attitude observer given by

$$\begin{aligned} \dot{\hat{\boldsymbol{x}}}_2(t) &= -\mathbf{S}_3\left(\boldsymbol{\omega}_m(t) - \mathbf{b}_{\boldsymbol{\omega}}(t)\right)\hat{\boldsymbol{x}}_2(t) \\ &\quad + \mathbf{C}_2^T \mathbf{Q}^{-1}\left[\mathbf{q}(t) - \mathbf{C}_2\hat{\boldsymbol{x}}_2(t)\right], \end{aligned} \tag{21}$$

where $\mathbf{Q} = \mathbf{Q}^T \in \mathbb{R}^{{}^N_2C\,{}^M_2C \times {}^N_2C\,{}^M_2C}$ is a positive definite matrix, and define the error variable $\tilde{\boldsymbol{x}}_2(t) = \boldsymbol{x}_2(t) - \hat{\boldsymbol{x}}_2(t)$. Then, the observer error dynamics are given by

$$\dot{\tilde{\boldsymbol{x}}}_2(t) = \boldsymbol{\mathcal{A}}_2(t)\tilde{\boldsymbol{x}}_2(t), \tag{22}$$

where

$$\boldsymbol{\mathcal{A}}_2(t) := -\left[\mathbf{S}_3\left(\boldsymbol{\omega}_m(t) - \mathbf{b}_{\boldsymbol{\omega}}(t)\right) + \mathbf{C}_2^T\mathbf{Q}^{-1}\mathbf{C}_2\right].$$

The following theorem is the main result of this section.

Theorem 2: Suppose that the rate gyro bias is known and consider the attitude observer (21), where $\mathbf{Q} \succ \mathbf{0}$ is a design parameter. Then, under Assumptions 1 and 2, the origin of the observer error dynamics (22) is a globally exponentially stable equilibrium point.

Proof: The proof follows by considering the Lyapunov candidate function $V_2(t) := \frac{1}{2}\left\|\tilde{\boldsymbol{x}}_2(t)\right\|^2$. It is similar to that of [15, Theorem 2] and therefore it is omitted. The only difference is, in fact, in the definition of $\mathbf{C}_2$, which is full rank, the only requirement for the proof. ■

C. Cascade observer

This section presents the overall cascade observer and its stability analysis, whose idea is to feed the attitude observer proposed in Section III-B with the bias estimate provided by the bias observer proposed in Section III-A. The bias observer remains the same, given by (15) and (16), whereas the attitude observer is now written as

$$\begin{aligned} \dot{\hat{\boldsymbol{x}}}_2(t) &= -\mathbf{S}_3\left(\boldsymbol{\omega}_m(t) - \hat{\mathbf{b}}_{\boldsymbol{\omega}}(t)\right)\hat{\boldsymbol{x}}_2(t) \\ &\quad + \mathbf{C}_2^T \mathbf{Q}^{-1}\left[\mathbf{q}(t) - \mathbf{C}_2\hat{\boldsymbol{x}}_2(t)\right]. \end{aligned} \tag{23}$$

The error dynamics corresponding to the bias observer are the same and therefore Theorem 1 applies. Evidently, the use of an estimate of the bias instead of the bias itself in the attitude observer introduces an error, and the stability of the system must be further examined. In this situation, the error dynamics of the cascade observer can be written as

$$\begin{cases} \dot{\tilde{\boldsymbol{x}}}_1(t) = \boldsymbol{\mathcal{A}}_1(t)\tilde{\boldsymbol{x}}_1(t) \\ \dot{\tilde{\boldsymbol{x}}}_2(t) = \left[\boldsymbol{\mathcal{A}}_2(t) - \mathbf{S}_3\left(\tilde{\mathbf{b}}_{\boldsymbol{\omega}}(t)\right)\right]\tilde{\boldsymbol{x}}_2(t) + \mathbf{u}_2(t), \end{cases} \tag{24}$$

where $\mathbf{u}_2(t) := \mathbf{S}_3\left(\tilde{\mathbf{b}}_{\boldsymbol{\omega}}(t)\right)\boldsymbol{x}_2(t)$.

The following theorem is the main result of the paper.

Theorem 3: Consider the cascade attitude observer given by (15), (16), and (23). Then, in the conditions of Theorem 1 and Theorem 2, the origin of the observer error dynamics (24) is a globally exponentially stable equilibrium point.

Proof: The proof follows exactly the same steps of [15, Theorem 3] and therefore it is omitted, even though the specific system dynamics are different. It is omitted due to space limitations. ■

IV. Simulation results

This section presents some simulation results in order to give an idea of the estimation performance achieved with the proposed solution. The 3-D kinematic model of an underwater vehicle is employed in the simulations as the proposed observer relies solely on the vehicle kinematics, which are exact. Therefore, the proposed solution applies to any underwater vehicle, regardless of the particular dynamics. The trajectory described by the vehicle is shown in Fig. 2. The LBL configuration is composed

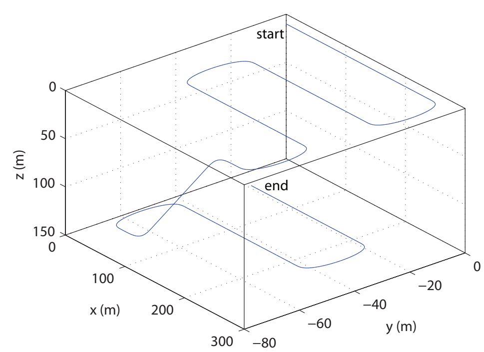


Fig. 2. Trajectory described by the vehicle

of 4 acoustic transponders and their inertial positions are $\mathbf{s}_1 = \begin{bmatrix} 1000 & 0 & 0 \end{bmatrix}$(m), $\mathbf{s}_2 = \begin{bmatrix} 0 & 1000 & 0 \end{bmatrix}$(m), $\mathbf{s}_3 = \begin{bmatrix} 1000 & 1000 & 0 \end{bmatrix}$(m), $\mathbf{s}_4 = \begin{bmatrix} 0 & 0 & 100 \end{bmatrix}$(m), while the positions of the USBL array receivers, in body-fixed coordinates, are $\mathbf{a}_1 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$(m), $\mathbf{a}_2 = \begin{bmatrix} 0 & 0.3 & 0 \end{bmatrix}$(m), $\mathbf{a}_3 = \begin{bmatrix} 0.20 & 0.15 & 0.15 \end{bmatrix}$(m), $\mathbf{a}_4 = \begin{bmatrix} 0.20 & 0.156 & -0.15 \end{bmatrix}$(m), hence both Assumptions 1 and 2 are satisfied.

Sensor noise was considered for all sensors. In particular, the LBL range measurements and the USBL range differences of arrival are assumed to be corrupted by additive uncorrelated zero-mean white Gaussian noise, with standard deviations of $1\,\mathrm{m}$ and $6 \times 10^{-3}\,\mathrm{m}$, respectively. The angular velocity measurements are also assumed to be perturbed by additive, zero mean, white Gaussian noise, with standard deviation of 0.05 °/s.

The observer parameters were chosen as $\alpha\left(m,n,i,j\right) = 0.1$, $\beta\left(m,n,i,j\right) = 5 \times 10^{-8}$ for all $(m,n,i,j) \in \mathcal{C}_\mathbf{s} \times \mathcal{C}_\mathbf{a}$, and $\mathbf{Q} = 10^4\mathbf{I}$. The initial condition of the rate gyro bias observer was set to zero, while $\mathbf{R}\left(0\right) = \mathbf{blkdiag}\left(-1,-1,1\right)$.

The convergence of the observer error is very fast, as it is possible to observe from the evolution of the errors of the components of the rotation matrix $\mathbf{R}(t)$ and the rate gyro bias error, which are depicted in Figs. 3 and 4, respectively. Although it is not shown here due to space limitations, the rate gyro bias error is confined, in steady-state, to a very tight interval, below 0.01 °/s.

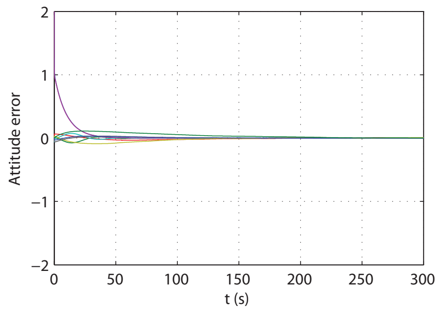


Fig. 3. Initial convergence of the attitude angle error

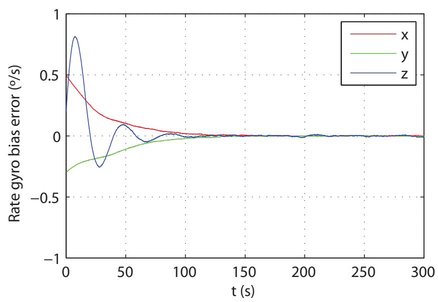


Fig. 4. Initial convergence of the rate gyro bias error

In order to evaluate the performance of the attitude observer, and for the purpose of performance evaluation only, an additional error variable is defined as $\tilde{\mathbf{R}}_p(t) = \mathbf{R}^T(t)\hat{\mathbf{R}}(t)$, which corresponds to the rotation matrix error. Using the Euler angle-axis representation for this new error variable,

$$\begin{aligned}\tilde{\mathbf{R}}_p(t) &= \mathbf{I}\cos\left(\tilde{\theta}(t)\right) + \left[1 - \cos\left(\tilde{\theta}(t)\right)\right]\tilde{\mathbf{d}}(t)\tilde{\mathbf{d}}^T(t) \\ &\quad -\mathbf{S}\left(\tilde{\mathbf{d}}(t)\right)\sin\left(\tilde{\theta}(t)\right),\end{aligned}$$

where $0 \leq \tilde{\theta}(t) \leq \pi$ and $\tilde{\mathbf{d}}(t) \in \mathbb{R}^3$, $\left\|\tilde{\mathbf{d}}(t)\right\| = 1$, are the angle and axis that represent the rotation error, the performance of the observer is identified with the evolution of $\tilde{\theta}$. After the initial transients fade out, the resulting angle mean error is 0.19 °.

V. Conclusions

This paper presented a novel attitude observer for underwater vehicles based on a combined LBL/USBL acoustic positioning system. In the envisioned solution, the range and range differences of arrival are directly used in the observer, as measured by the USBL/LBL. No linearizations are carried out whatsoever and the resulting error dynamics are globally exponentially stable. The design is computationally efficient and preliminary simulations results are shown that illustrate the achievable performance. Future work includes the analysis of the multi-rate case, the treatment of measurement outages and outliers, and comparison with other solutions.

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