Systems & Control Letters 62 (2013) 1065–1072

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Globally exponentially stable filters for source localization and navigation aided by direction measurements

Pedro Batista [a,b,*], Carlos Silvestre [a,c], Paulo Oliveira [a,d]

[a] Institute for Systems and Robotics, Laboratory of Robotics and Systems in Engineering and Science, Portugal
[b] Instituto Superior Técnico, Universidade de Lisboa, Portugal
[c] Department of Electrical and Computer Engineering, Faculty of Science and Technology, University of Macau, Macao
[d] Department of Mechanical Engineering, Instituto Superior Técnico, Universidade de Lisboa, Portugal

ARTICLE INFO

Article history:
Received 21 October 2011
Received in revised form
20 March 2013
Accepted 28 July 2013
Available online 23 September 2013

Keywords:
Single beacon navigation
Source localization
Observability of nonlinear systems
3-D linear motion kinematics
Navigation systems

ABSTRACT

This paper presents a set of filters with globally exponentially stable error dynamics for source localization and navigation, in 3-D, based on direction measurements from the agent (or vehicle) to the source, in addition to relative velocity readings of the agent. Both the source and the agent are allowed to have constant unknown drift velocities and the relative drift velocity is also explicitly estimated. The observability of the system is studied and realistic simulation results are presented, in the presence of measurement noise, that illustrate the performance of the achieved solutions. Comparison results with the Extended Kalman Filter are also provided and similar performances are achieved.

1. Introduction

The problem of source localization has been the subject of intensive research in recent years. Roughly speaking, an agent has access to a set of measurements and aims to estimate the position of a source. The set of measurements depends on the environment in which the operation occurs and the mission scenario itself. Previous work in the field can be found in [1], where the authors propose a localization algorithm based on the range to the source and the inertial position of the agent. Global exponential stability (GES) is achieved under a persistent excitation condition and the analysis is extended to the case of a non-stationary source, where it is shown that it is possible to achieve tracking up to some bounded error. In [2] the same problem was addressed considering, in addition to range readings to the source, relative velocity readings of the agent. The observability of the system was assessed, including also relative drift velocities, and filtering solutions were proposed with globally asymptotically stable error dynamics. More recently, in [3], the same problem was addressed, in 2-D, based on bearing measurements, in addition to the trajectory of the agent. The estimation error dynamics were shown to be GES under an appropriate persistent excitation condition and a circumnavigation control law was also proposed. Earlier work on the observability issues of target motion analysis based on angle readings, in 2-D, can be found in [4], which was later extended to 3-D in [5]. The specific observability criteria thereby derived resort to complicated nonlinear differential equations and some tedious mathematics are needed for the solution, giving conditions that are necessary for system observability. Another related framework in the domain of target motion analysis (TMA) can be found in [6], where frequency measurements are also included. This topic was further studied in [7], where Cramer–Rao analysis revealed the parametric dependencies of TMA with angle-only tracking and angle/frequency tracking, giving also an idea of the increase in estimation accuracy using the later.

Parallel to the topic of source localization based on range or bearing measurements is the topic of navigation aided by these sensors. Previous work by the authors with range measurements can be found in [8], where acceleration readings were also considered. The observability of the system was assessed and conditions were derived that guarantee globally asymptotically stable error dynamics. In [9] a similar design was proposed with two vehicles working in tandem considering relative velocity drifts. Globally asymptotically stable error dynamics were also shown under appropriate observability conditions. In [10] the authors deal with the problem of underwater navigation in the presence of unknown

* Correspondence to: Institute for Systems and Robotics, Instituto Superior Técnico, Universidade de Lisboa, Portugal.
E-mail addresses: pbatista@isr.ist.utl.pt (P. Batista), cjs@isr.ist.utl.pt (C. Silvestre), pjcro@isr.ist.utl.pt (P. Oliveira).

0167-6911/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.sysconle.2013.07.010

currents based on range measurements to a single beacon. An observability analysis is presented based on the linearization of the nonlinear system which yields local results. Based on the linearized system dynamics, a Luenberger observer is introduced but in practice an Extended Kalman filter (EKF) is implemented, with no warranties of global asymptotic stability. More recently, the same problem has been studied in [11,12], where EKFs have been extensively used to solve the navigation problem based on single beacon range measurements. The problem of localization of a mobile robot using bearing measurements was also addressed in [13], where a nonlinear transformation of the measurement equation into a higher dimensional space is performed. This has allowed to obtain tight, possibly complex-shaped, bounding sets for the feasible states in a closed-form representation.

This paper addresses the problem of navigation/source localization based on direction measurements to a single source in the presence of unknown constant drifts. The observability of the system is studied and Kalman filters with GES error dynamics are proposed, without system linearizations and yielding performances comparable to those of the Extended Kalman Filter but with GES guarantees. Central to the design is the augmentation of the system state, which allowed to consider linear time-varying (LTV) system dynamics. The observability conditions have clear physical meaning and they are directly related to the motion of the agent/vehicle, hence useful for motion planning and control so that the system is observable. Preliminary work by the authors can be found in [14]. In addition to more detailed explanations and further discussion of issues such as the physical meaning of the observability conditions, the present paper acknowledges GES error dynamics and includes an additional solution for navigation that does not require the knowledge of the source velocity.

1.1. Notation

Throughout the paper the symbol $\mathbf{0}$ denotes a matrix (or vector) of zeros and $\mathbf{I}$ an identity matrix, both of appropriate dimensions. A block diagonal matrix is represented as $\operatorname{diag}(\mathbf{A}_1, \ldots, \mathbf{A}_n)$ and the set of unit vectors on $\mathbb{R}^3$ is denoted by $S(2)$. Finally, $\delta(t)$ corresponds to the Dirac delta function.

2. Problem statement

2.1. Source localization

Let $\mathbf{p}(t) \in \mathbb{R}^3$ denote the position of a point-mass agent, in inertial coordinates, moving in a scenario where there is a source whose position, in inertial coordinates, is denoted by $\mathbf{s}(t) \in \mathbb{R}^3$. Suppose that the source is moving with constant unknown velocity $\mathbf{v}_s(t) \in \mathbb{R}^3$ relative to the inertial frame, which gives $\dot{\mathbf{s}}(t) = \mathbf{v}_s(t)$ and $\dot{\mathbf{v}}_s(t) = \mathbf{0}$, while the linear motion kinematics of the agent are given by $\dot{\mathbf{p}}(t) = \mathbf{v}_c(t) + \mathbf{v}_r(t)$ and $\dot{\mathbf{v}}_c(t) = \mathbf{0}$, where $\mathbf{v}_c(t) \in \mathbb{R}^3$ is a constant unknown drift velocity of the agent and $\mathbf{v}_r(t) \in \mathbb{R}^3$ is a known input. In the context of the EU project TRIDENT, the source is an Autonomous Surface Craft (ASC) and the agent an Autonomous Underwater Vehicle (AUV). The ASC is moving with constant unknown velocity $\mathbf{v}_s(t)$ and the AUV is moving with velocity relative to the water $\mathbf{v}_r(t)$, as given by a Doppler Velocity Log (DVL), in the presence of constant unknown ocean currents with velocity $\mathbf{v}_c(t)$. Further consider that the agent measures the direction to the source

$$\mathbf{d}(t) = \frac{\mathbf{r}(t)}{\|\mathbf{r}(t)\|} \in S(2), \tag{1}$$

with $\mathbf{r}(t) := \mathbf{s}(t) - \mathbf{p}(t) \in \mathbb{R}^3$. The problem of source localization considered here is that of estimating the position of the source relative to the agent, $\mathbf{r}(t)$, and the relative drift velocity $\mathbf{v}_{sa}(t) := \mathbf{v}_s(t) - \mathbf{v}_c(t) \in \mathbb{R}^3$, given direction and relative velocity readings, $\mathbf{d}(t)$ and $\mathbf{v}_r(t)$, respectively. The corresponding system dynamics are given by

$$\begin{cases} \dot{\mathbf{r}}(t) = \mathbf{v}_{sa}(t) - \mathbf{v}_r(t) \\ \dot{\mathbf{v}}_{sa}(t) = \mathbf{0} \\ \mathbf{d}(t) = \dfrac{\mathbf{r}(t)}{\|\mathbf{r}(t)\|}. \end{cases}$$

The following assumption is required in the sequel.

Assumption 1. The relative velocity is continuous and continuously differentiable. Moreover, both $\mathbf{v}_r(t)$ and $\dot{\mathbf{v}}_r(t)$ are norm-bounded.

This is a mild assumption with clear physical interpretation as the actuation systems of agents or vehicles limit the available force and torque, which implies upper bounds on the velocities and accelerations. In this paper it allows to consider that both $\dot{\mathbf{d}}(t)$ and $\ddot{\mathbf{d}}(t)$ are norm-bounded. The values of the bounds are not required.

2.2. Navigation

In the context of the EU project TRIDENT, an ASC and an AUV work in close cooperation in order to achieve a certain goal. Assume that the ASC (the source) transmits its inertial position $\mathbf{s}(t)$ and velocity $\mathbf{v}_s(t)$ to the AUV (the agent). In this framework, the goal of the AUV (the agent) is now to determine its own position in inertial coordinates $\mathbf{p}(t)$, as well as its drift velocity $\mathbf{v}_c(t)$, given the information provided by the ASC (the source), the relative velocity readings $\mathbf{v}_r(t)$, and the direction measurements $\mathbf{d}(t)$. In this framework $\mathbf{v}_s(t)$ is no longer required to be constant and the system dynamics are given by

$$\begin{cases} \dot{\mathbf{p}}(t) = \mathbf{v}_c(t) + \mathbf{v}_r(t) \\ \dot{\mathbf{v}}_c(t) = \mathbf{0} \\ \mathbf{d}(t) = \dfrac{\mathbf{s}(t) - \mathbf{p}(t)}{\|\mathbf{s}(t) - \mathbf{p}(t)\|}. \end{cases} \tag{2}$$

3. Source localization filter design

3.1. System dynamics

In order to derive an augmented linear time-varying system for source localization, consider the system states $\mathbf{x}_1(t) := \mathbf{r}(t)$, $\mathbf{x}_2(t) := \mathbf{v}_{sa}(t)$, and $x_3(t) := \|\mathbf{r}(t)\|$ and define the state vector $\mathbf{x}(t) := \left[\mathbf{x}_1^T(t)\,\mathbf{x}_2^T(t)\,x_3(t)\right]^T \in \mathbb{R}^7$. From (1) it follows that $\mathbf{x}_1(t) - x_3(t)\mathbf{d}(t) = \mathbf{0}$ for all t. Then, the system dynamics are given by the LTV system

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t), \end{cases} \tag{3}$$

where

$$\mathbf{A}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{d}^T(t) & 0 \end{bmatrix} \in \mathbb{R}^{7\times 7},$$

$$\mathbf{B}(t) = \begin{bmatrix} -\mathbf{I} \\ \mathbf{0} \\ -\mathbf{d}^T(t) \end{bmatrix} \in \mathbb{R}^{7\times 3},$$

$\mathbf{C}(t) = [\mathbf{I}\ \mathbf{0}\ -\mathbf{d}(t)] \in \mathbb{R}^{3\times 7}$, and $\mathbf{u}(t) = \mathbf{v}_r(t)$.

3.2. Observability analysis

The observability of the problem of source localization with relative velocity readings and direction measurements is studied in this section. The following proposition (Proposition 4.2, [15]) is useful in the sequel.

Proposition 1. *Let* $\mathbf{f}(t) : \left[t_0, t_f\right] \subset \mathbb{R} \to \mathbb{R}^n$ *be a continuous and i-times continuously differentiable function on* $\mathcal{I} := \left[t_0, t_f\right]$, $T := t_f - t_0 > 0$, *and such that* $\mathbf{f}(t_0) = \dot{\mathbf{f}}(t_0) = \cdots = \mathbf{f}^{(i-1)}(t_0) = \mathbf{0}$. *Further assume that there exists a nonnegative constant* C *such that* $\left\|\mathbf{f}^{(i+1)}(t)\right\| \leq C$ *for all* $t \in \mathcal{I}$. *If there exist* $\alpha > 0$ *and* $t_1 \in \mathcal{I}$ *such that* $\left\|\mathbf{f}^{(i)}(t_1)\right\| \geq \alpha$, *then there exist* $0 < \delta \leq T$ *and* $\beta > 0$ *such that* $\|\mathbf{f}(t_0 + \delta)\| \geq \beta$.

The following theorem characterizes the observability of the LTV system (3).

Theorem 1. *The LTV system* (3) *is observable on* $\mathcal{I} := \left[t_0, t_f\right]$ *if and only if the unit vector* $\mathbf{d}(t)$ *is not constant on* $\mathcal{I}$ *or, equivalently,*

$$\exists_{t_1 \in \mathcal{I}} \quad \mathbf{d}^T(t_0)\,\mathbf{d}(t_1) < 1. \tag{4}$$

Proof. The observability Gramian associated with the pair $(\mathbf{A}(t), \mathbf{C}(t))$ on $\mathcal{I}$ is given by

$$\mathcal{W}\left(t_0, t_f\right) = \int_{t_0}^{t_f} \boldsymbol{\phi}^T(\tau, t)\,\mathbf{C}^T(\tau)\mathbf{C}(\tau)\boldsymbol{\phi}(\tau, t)\,\mathrm{d}\tau,$$

where $\boldsymbol{\phi}(t, t_0)$ is the transition matrix associated with $\mathbf{A}(t)$,

$$\boldsymbol{\phi}(t, t_0) = \begin{bmatrix} \mathbf{I} & (t - t_0)\,\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \displaystyle\int_{t_0}^{t} \mathbf{d}^T(\tau)\,\mathrm{d}\tau & 1 \end{bmatrix} \in \mathbb{R}^{7\times 7}.$$

Let $\mathbf{c} = \begin{bmatrix}\mathbf{c}_1^T & \mathbf{c}_2^T & c_3\end{bmatrix}^T \in \mathbb{R}^7$, $\mathbf{c}_i \in \mathbb{R}^3$, $i = 1, 2$, $c_3 \in \mathbb{R}$, be a unit vector, i.e., $\|\mathbf{c}\| = 1$. Then,

$$\mathbf{c}^T \mathcal{W}\left(t_0, t_f\right)\mathbf{c} = \int_{t_0}^{t_f} \|\mathbf{f}(\tau)\|^2\,\mathrm{d}\tau$$

for all $\|\mathbf{c}\| = 1$, where

$$\mathbf{f}(\tau) = \mathbf{c}_1 + \left[(\tau - t_0)\,\mathbf{I} - \mathbf{d}(\tau)\int_{t_0}^{\tau} \mathbf{d}^T(\sigma)\mathrm{d}\sigma\right]\mathbf{c}_2 - c_3\mathbf{d}(\tau)$$

for all $\tau \in \mathcal{I}$. The first two derivatives of $\mathbf{f}(\tau)$ are given by

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{f}(\tau) = \left[\mathbf{I} - \mathbf{d}(\tau)\mathbf{d}^T(\tau) - \dot{\mathbf{d}}(\tau)\int_{t_0}^{\tau} \mathbf{d}^T(\sigma)\mathrm{d}\sigma\right]\mathbf{c}_2 - c_3\dot{\mathbf{d}}(\tau)$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\mathbf{f}(\tau) = \Bigg[-2\dot{\mathbf{d}}(\tau)\mathbf{d}^T(\tau) - \mathbf{d}(\tau)\dot{\mathbf{d}}^T(\tau) - \ddot{\mathbf{d}}(\tau)\int_{t_0}^{\tau} \mathbf{d}^T(\sigma)\mathrm{d}\sigma\Bigg]\mathbf{c}_2 - c_3\ddot{\mathbf{d}}(\tau)$$

for all $\tau \in \mathcal{I}$. Notice that, under Assumption 1, both derivatives are norm-bounded, from above, on $\mathcal{I}$.

The proof of necessity follows by contraposition. Suppose that (4) is not verified. Then, the unit vector $\mathbf{d}(t)$ is constant on $\mathcal{I}$, i.e., $\mathbf{d}(t) = \mathbf{d}(t_0)$ for all $t \in \mathcal{I}$. Let $\mathbf{c}_1 = \frac{\sqrt{2}}{2}\mathbf{d}(t_0)$, $\mathbf{c}_2 = \mathbf{0}$, and $c_3 = \frac{\sqrt{2}}{2}$. Then, it follows that $\mathbf{f}(\tau) = \frac{\sqrt{2}}{2}\mathbf{d}(t_0) - \frac{\sqrt{2}}{2}\mathbf{d}(\tau) = \mathbf{0}$ for all $\tau \in \mathcal{I}$, which in turn allows to conclude that the observability Gramian $\mathcal{W}\left(t_0, t_f\right)$ is not invertible and the LTV system (3) is not observable on $\mathcal{I}$. Consequently, if the LTV system (3) is observable on $\mathcal{I}$, it follows that (4) is true.

To show that (4) is also a sufficient condition, suppose first that $c_3 \neq 0$. Then, if $\mathbf{c}_1 \neq c_3\mathbf{d}(t_0)$, it follows that $\|\mathbf{f}(t_0)\| > 0$ and, from Proposition 1, it must be $\mathbf{c}^T \mathcal{W}\left(t_0, t_f\right)\mathbf{c} > 0$. Consider now $\mathbf{c}_1 = c_3\mathbf{d}(t_0)$, with $c_3 \neq 0$. In this case, $\mathbf{f}(t_0) = \mathbf{0}$ and

$$\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{f}(\tau)\right|_{\tau=t_0} = \left[\mathbf{I} - \mathbf{d}(t_0)\,\mathbf{d}^T(t_0)\right]\mathbf{c}_2 - c_3\dot{\mathbf{d}}(t_0).$$

If $\left\|\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{f}(\tau)\right|_{\tau=t_0}\right\| > 0$, it follows, using Proposition 1 twice, that $\mathbf{c}^T \mathcal{W}\left(t_0, t_f\right)\mathbf{c} > 0$. Otherwise, if $\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{f}(\tau)\right|_{\tau=t_0} = \mathbf{0}$, two cases may be considered: (i) if $\dot{\mathbf{d}}(t_0) = \mathbf{0}$, it may be $\mathbf{c}_2 = \mathbf{0}$ or $\mathbf{c}_2 = c_2\mathbf{d}(t_0)$ for some scalar c_2; or (ii) if $\dot{\mathbf{d}}(t_0) \neq \mathbf{0}$, it must be $\mathbf{c}_2 = c_3\dot{\mathbf{d}}(t_0)$, where it is used the fact that $\mathbf{d}^T(t)\dot{\mathbf{d}}(t) = 0$ for all t. Evaluating $\mathbf{f}(\tau)$ at $\tau = t_1$, when $\mathbf{c}_2 = \mathbf{0}$, yields $\mathbf{f}(t_1) = c_3\mathbf{d}(t_0) - c_3\mathbf{d}(t_1)$ which has a positive norm if (4) is true. As such, it follows from Proposition 1 that $\mathbf{c}^T \mathcal{W}\left(t_0, t_f\right)\mathbf{c} > 0$ for $\mathbf{c}_1 = c_3\mathbf{d}(t_0)$, $\mathbf{c}_2 = \mathbf{0}$, $c_3 \neq 0$. If $\mathbf{c}_2 = c_2\mathbf{d}(t_0)$, $\mathbf{f}(t_1)$ reads as

$$\mathbf{f}(t_1) = \left[c_3 + c_2(t_1 - t_0)\right]\mathbf{d}(t_0) - \left[c_3 + c_2\int_{t_0}^{t_1} \mathbf{d}^T(\sigma)\mathbf{d}(t_0)\,\mathrm{d}\sigma\right]\mathbf{d}(t_1).$$

If (4) is true, and as $\mathbf{d}(t)$ is a continuous function of time, it must be $\int_{t_0}^{t_1} \mathbf{d}^T(\sigma)\mathbf{d}(t_0)\,\mathrm{d}\sigma \neq t_1 - t_0$, which allows to conclude that $\|\mathbf{f}(t_1)\| > 0$. Hence, using Proposition 1, $\mathbf{c}^T \mathcal{W}\left(t_0, t_f\right)\mathbf{c} > 0$ for $\mathbf{c}_1 = c_3\mathbf{d}(t_0)$, $\mathbf{c}_2 = c_2\mathbf{d}(t_0)$, $c_2 \neq 0$, $c_3 \neq 0$. If $\mathbf{c}_2 = c_3\dot{\mathbf{d}}(t_0)$, with $\dot{\mathbf{d}}(t_0) \neq \mathbf{0}$ and $\mathbf{c}_1 = c_3\mathbf{d}(t_0)$, $c_3 \neq 0$, there exists $\epsilon > 0$ such that

$$\mathbf{f}(t_0 + \epsilon) = c_3\mathbf{d}(t_0) + c_3\epsilon\dot{\mathbf{d}}(t_0) - c_3\left[1 + \int_{t_0}^{t_0+\epsilon} \mathbf{d}^T(\sigma)\dot{\mathbf{d}}(t_0)\,\mathrm{d}\sigma\right]\mathbf{d}(t_0 + \epsilon),$$

where $\mathbf{d}(t_0 + \epsilon)$ cannot be expressed as a linear combination of $\mathbf{d}(t_0)$ and $\dot{\mathbf{d}}(t_0)$. As such, $\|\mathbf{f}(t_0 + \epsilon)\| > 0$ and, using Proposition 1, $\mathbf{c}^T \mathcal{W}\left(t_0, t_f\right)\mathbf{c} > 0$ for $\mathbf{c}_1 = c_3\mathbf{d}(t_0)$, $\mathbf{c}_2 = c_3\dot{\mathbf{d}}(t_0)$, $c_3 \neq 0$. This allows to conclude, so far, that if $c_3 \neq 0$, $\mathbf{c}^T \mathcal{W}\left(t_0, t_f\right)\mathbf{c} > 0$. It remains to see what happens when $c_3 = 0$. If $\mathbf{c}_1 \neq \mathbf{0}$, it turns out that $\|\mathbf{f}(t_0)\| > 0$ and again, using Proposition 1, it must be $\mathbf{c}^T \mathcal{W}\left(t_0, t_f\right)\mathbf{c} > 0$ for $\mathbf{c}_1 \neq \mathbf{0}$, $c_3 = 0$. On the other hand, if $\mathbf{c}_1 = \mathbf{0}$, $c_3 = 0$, it follows that $\mathbf{f}(t_0) = \mathbf{0}$ and

$$\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{f}(\tau)\right|_{\tau=t_0} = \left[\mathbf{I} - \mathbf{d}(t_0)\,\mathbf{d}^T(t_0)\right]\mathbf{c}_2.$$

Now, if $\mathbf{c}_2 \neq \pm\mathbf{d}(t_0)$, it follows that

$$\left\|\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{f}(\tau)\right|_{\tau=t_0}\right\| > 0$$

and, using Proposition 1 twice, it must be $\mathbf{c}^T \mathcal{W}\left(t_0, t_f\right)\mathbf{c} > 0$ for $\mathbf{c}_1 = \mathbf{0}$, $\mathbf{c}_2 \neq \pm\mathbf{d}(t_0)$, $c_3 = 0$. Finally, if $\mathbf{c}_2 = \pm\mathbf{d}(t_0)$, with $\mathbf{c}_1 = \mathbf{0}$ and $c_3 = 0$, it follows that

$$\mathbf{f}(t_1) = \pm(t_1 - t_0)\,\mathbf{d}(t_0) \mp \int_{t_0}^{\tau} \mathbf{d}^T(\sigma)\mathbf{d}(t_0)\,\mathrm{d}\sigma\,\mathbf{d}(t_1),$$

which has a positive norm. Again, using Proposition 1, it follows that $\mathbf{c}^T \mathcal{W}\left(t_0, t_f\right)\mathbf{c} > 0$ for $\mathbf{c}_1 = \mathbf{0}$, $\mathbf{c}_2 = \pm\mathbf{d}(t_0)$, $c_3 = 0$. But this concludes the proof, as it is shown that $\mathbf{c}^T \mathcal{W}\left(t_0, t_f\right)\mathbf{c} > 0$ for all $\|\mathbf{c}\| = 1$, which means that the observability Gramian in invertible and as such (3) is observable. □

Before proceeding, it is important to remark that there is nothing in (3) imposing the nonlinear restriction $\|\mathbf{x}_1(t)\| = x_3(t) = \|\mathbf{r}(t)\|$. This is true, by construction, if it is satisfied for $t = t_0$. The following theorem addresses this issue.

Theorem 2. *Under the hypothesis of Theorem 1, the initial condition of the LTV (3) corresponds to the initial condition of the original nonlinear system, i.e.,*

$$\begin{cases} \mathbf{x}_1(t_0) = \mathbf{r}(t_0) \\ \mathbf{x}_2(t_0) = \mathbf{v}_{sa}(t_0) \\ x_3(t_0) = \|\mathbf{r}(t_0)\|. \end{cases} \tag{5}$$

Proof. Under the terms of Theorem 1, the initial condition of the LTV system (3) is uniquely determined by the corresponding system output and input. The proof follows by showing that (5) explains the system output. As the initial condition is uniquely determined, if (5) explains the output of the system, it must correspond to the initial condition. The output of the LTV system (3) is given by

$$\mathbf{y}(t) = \mathbf{x}_1(t_0) + (t-t_0)\mathbf{x}_2(t_0) - \int_{t_0}^{t} \mathbf{u}(\tau)d\tau - x_3(t_0)\mathbf{d}(t) - \int_{t_0}^{t} [\mathbf{x}_2(t_0) - \mathbf{u}(\tau)]^T \mathbf{d}(\tau)d\tau\mathbf{d}(t) = \mathbf{0} \tag{6}$$

for all $t \in \mathcal{I}$, $\mathcal{I} = [t_0, t_f]$. Substituting (5) in (6) gives

$$\mathbf{y}(t) = \mathbf{r}(t_0) - \|\mathbf{r}(t_0)\|\mathbf{d}(t) + \int_{t_0}^{t} [\mathbf{v}_{sa}(t_0) - \mathbf{u}(\tau)]d\tau - \int_{t_0}^{t} [\mathbf{v}_{sa}(t_0) - \mathbf{u}(\tau)]^T \mathbf{d}(\tau)d\tau\mathbf{d}(t). \tag{7}$$

It remains only to show that (7) is null for all $t \in \mathcal{I}$. Substituting $t = t_0$ in (7) yields $\mathbf{y}(t_0) = \mathbf{0}$. The time derivative of (7) is given by

$$\dot{\mathbf{y}}(t) = -\left[\|\mathbf{r}(t_0)\| + \int_{t_0}^{t} [\mathbf{v}_{sa}(t_0) - \mathbf{u}(\tau)]^T \mathbf{d}(\tau)d\tau\right]\dot{\mathbf{d}}(t) + [\mathbf{v}_{sa}(t_0) - \mathbf{u}(t)] - [\mathbf{v}_{sa}(t_0) - \mathbf{u}(t)]^T \mathbf{d}(t)\mathbf{d}(t). \tag{8}$$

As $\mathbf{v}_{sa}(t)$ is constant, it is possible to rewrite (8) as

$$\dot{\mathbf{y}}(t) = -\left[\|\mathbf{r}(t_0)\| + \int_{t_0}^{t} [\mathbf{v}_{sa}(\tau) - \mathbf{u}(\tau)]^T \mathbf{d}(\tau)d\tau\right]\dot{\mathbf{d}}(t) + [\mathbf{v}_{sa}(t) - \mathbf{u}(t)] - [\mathbf{v}_{sa}(t) - \mathbf{u}(t)]^T \mathbf{d}(t)\mathbf{d}(t). \tag{9}$$

Using the derivative $\frac{d}{dt}\|\mathbf{r}(t)\| = [\mathbf{v}_{sa}(t) - \mathbf{u}(t)]^T \mathbf{d}(t)$, allows to write

$$\|\mathbf{r}(t)\| = \|\mathbf{r}(t_0)\| + \int_{t_0}^{t} [\mathbf{v}_{sa}(\tau) - \mathbf{u}(\tau)]^T \mathbf{d}(\tau)d\tau. \tag{10}$$

On the other hand, the time derivative of (1) is given by

$$\dot{\mathbf{d}}(t) = \frac{[\mathbf{v}_{sa}(t) - \mathbf{u}(t)] - [\mathbf{v}_{sa}(t) - \mathbf{u}(t)]^T \mathbf{d}(t)\mathbf{d}(t)}{\|\mathbf{r}(t)\|}. \tag{11}$$

Substituting (10) and (11) in (9) gives $\dot{\mathbf{y}}(t) = \mathbf{0}$. This concludes the proof, as with $\mathbf{y}(t_0) = \mathbf{0}$ and $\dot{\mathbf{y}}(t) = \mathbf{0}$ it must be $\mathbf{y}(t) = \mathbf{0}$ for all $t \in \mathcal{I}$ and therefore (5) is true. □

In order to design GES observers (or filtering) solutions, stronger forms of observability are convenient. The following theorem addresses this issue.

Theorem 3. *The LTV system (3) is uniformly completely observable if and only if*

$$\exists_{\substack{\alpha>0 \\ \delta>0}} \forall_{t \geq t_0} \int_{t}^{t+\delta} \mathbf{d}^T(t)\mathbf{d}(\tau)d\tau \leq \delta(1-\alpha). \tag{12}$$

Proof. The proof of sufficiency follows similar steps to Theorem 1 considering uniformity bounds that stem from the persistent excitation condition (12). Therefore it is omitted. To show that (12) is also necessary, suppose that (12) does not hold. Then,

$$\forall_{\substack{\alpha>0 \\ \delta>0}} \exists_{t^* \geq t_0} \int_{t^*}^{t^*+\delta} \mathbf{d}^T(t^*)\mathbf{d}(t)dt > \delta(1-\alpha). \tag{13}$$

Let $\mathbf{c} = \begin{bmatrix} \frac{\sqrt{2}}{2}\mathbf{d}^T(t^*) & \mathbf{0} & \sqrt{2}/2 \end{bmatrix}^T \in \mathbb{R}^7$. Then,

$$\mathbf{c}^T \mathcal{W}(t^*, t^*+\delta)\mathbf{c} = \frac{1}{2}\int_{t^*}^{t^*+\delta} \|\mathbf{d}(t^*) - \mathbf{d}(\tau)\|^2 d\tau = \frac{1}{2}\int_{t^*}^{t^*+\delta} \left[\|\mathbf{d}(t^*)\|^2 + \|\mathbf{d}(\tau)\|^2 - 2\mathbf{d}^T(t^*)\mathbf{d}(\tau)\right] d\tau. \tag{14}$$

As $\mathbf{d}(t)$ is a unit vector, it is possible to write (14) as

$$\mathbf{c}^T \mathcal{W}(t^*, t^*+\delta)\mathbf{c} = \delta - \int_{t^*}^{t^*+\delta} \mathbf{d}^T(t^*)\mathbf{d}(\tau)d\tau. \tag{15}$$

Using (13) in (15) allows to conclude that for all $\alpha > 0$ and $\delta > 0$ there exists time instant $t^* \geq t_0$ such that $\mathbf{c}^T \mathcal{W}(t^*, t^*+\delta)\mathbf{c} < \delta\alpha$, which means that the LTV system (3) is not uniformly completely observable. Therefore, if the LTV system (3) is uniformly completely observable, (12) is true. □

For observability over a fixed time interval Theorem 1 already provides sufficient insight: the system is observable if the direction measurements do not remain constant on that interval. For uniform complete observability the result provided by Theorem 3 is essentially an extension considering uniformity in time: the system is uniformly completely observable if it is possible to choose a fixed time interval length $\delta > 0$ such that, for all time intervals of length δ, there is a minimum variation in the direction measurements, uniformly in time, which is encoded by the positive constant α in (12).

3.3. Kalman filter

Section 3.1 introduced a LTV system for source localization and its observability was characterized in Section 3.2. In particular, it was shown that the LTV system (3) is uniformly completely observable if and only if an appropriate persistent excitation condition, (12), is satisfied. As such, the design of a Kalman filter, with globally exponentially stable error dynamics, follows naturally. An alternative observer with globally exponentially stable error dynamics could be devised using [16, Theorem 15.2]. Considering additive system disturbances and sensor noise, the system dynamics are given by

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{w}(t) \\ \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{n}(t), \end{cases}$$

where $\mathbf{w}(t) \in \mathbb{R}^7$ is zero-mean white Gaussian noise, with $E[\mathbf{w}(t)\mathbf{w}^T(t-\tau)] = \boldsymbol{\Xi}\delta(\tau)$, $\boldsymbol{\Xi} \succ \mathbf{0}$, $\mathbf{n}(t) \in \mathbb{R}^3$ is zero-mean white Gaussian noise, with $E[\mathbf{n}(t)\mathbf{n}^T(t-\tau)] = \boldsymbol{\Theta}\delta(\tau)$, $\boldsymbol{\Theta} \succ \mathbf{0}$, and $E[\mathbf{w}(t)\mathbf{n}^T(t-\tau)] = \mathbf{0}$. It is important to stress, however, that it is not possible to conclude that this is an optimal solution, as the actual system disturbances and sensor noise may not be additive. Nevertheless, the nominal filter error dynamics are globally exponentially stable if the LTV system is uniformly completely observable and controllable [17]. The design of the Kalman filter is well known and therefore it is omitted.

Remark 1. Even though the drift velocities are assumed, in nominal terms, as constant, it is possible to track slowly time-varying drift velocities (up to some error) by appropriate tuning of the corresponding state disturbance covariance design parameter of the Kalman filter.

4. Navigation filter design assuming known source velocity

This section presents a solution for navigation based on direction measurements similar to the solution for source localization proposed in Section 3. In order to derive an augmented linear time-varying system for navigation based on direction readings, define the system states $\mathbf{x}_1(t) := \mathbf{p}(t)$, $\mathbf{x}_2(t) := \mathbf{v}_c(t)$, and $x_3(t) = \|\mathbf{r}(t)\|$. From (1) it follows that $\mathbf{x}_1(t) + x_3(t)\mathbf{d}(t) = \mathbf{s}(t)$ for all t. Let $\mathbf{x}(t) = \left[\mathbf{x}_1^T(t)\, \mathbf{x}_2^T(t)\, x_3(t)\right]^T \in \mathbb{R}^7$. Then, the system dynamics are given by the LTV system

$$\begin{cases}\dot{\mathbf{x}}(t) = \mathcal{A}(t)\mathbf{x}(t) + \mathcal{B}(t)\mathbf{u}(t)\\ \mathbf{y}(t) = \mathcal{C}(t)\mathbf{x}(t),\end{cases} \tag{16}$$

where

$$\mathcal{A}(t) = \begin{bmatrix}\mathbf{0} & \mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & -\mathbf{d}^T(t) & 0\end{bmatrix} \in \mathbb{R}^{7\times 7},$$

$$\mathcal{B}(t) = \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\\ -\mathbf{d}^T(t) & \mathbf{d}^T(t)\end{bmatrix} \in \mathbb{R}^{7\times 6},$$

$\mathcal{C}(t) = [\mathbf{I}\ \mathbf{0}\ \mathbf{d}(t)] \in \mathbb{R}^{3\times 7}$, and $\mathbf{u}(t) = \left[\mathbf{v}_r^T(t) \quad \mathbf{v}_s^T(t)\right]^T \in \mathbb{R}^6$.

Next, consider the Lyapunov state transformation of the LTV system (16) given by $\mathbf{z}(t) = \operatorname{diag}(\mathbf{I}, \mathbf{I}, -1)\,\mathbf{x}(t)$, which preserves observability properties. The new system dynamics read as

$$\begin{cases}\dot{\mathbf{z}}(t) = \mathbf{A}(t)\mathbf{z}(t) + \operatorname{diag}(\mathbf{I}, \mathbf{I}, -1)\,\mathcal{B}(t)\mathbf{u}(t)\\ \mathbf{y}(t) = \mathbf{C}(t)\mathbf{z}(t).\end{cases}$$

Notice that the new system matrices $\mathbf{A}(t)$ and $\mathbf{C}(t)$ are those of the LTV system (3). This immediately allows to characterize the observability of the LTV system (16) with the following two theorems, as both systems are related by a Lyapunov state transformation [18].

Theorem 4. *The LTV system (16) is observable on $\mathcal{I} := \left[t_0, t_f\right]$ if and only if the unit vector $\mathbf{d}(t)$ is not constant on $\mathcal{I}$ or, equivalently, (4) is true.*

Theorem 5. *The LTV system (16) is uniformly completely observable if and only if (12) holds.*

It remains to see that, as in the solution for source localization, the initial condition of the LTV system, uniquely determined under the observability condition expressed in the previous theorems, matches the initial condition of the original system. This is shown in the following theorem.

Theorem 6. *Under the hypothesis of Theorem 4, the initial condition of the LTV (16) corresponds to the initial condition of the original nonlinear system, i.e.,*

$$\begin{cases}\mathbf{x}_1(t_0) = \mathbf{p}(t_0)\\ \mathbf{x}_2(t_0) = \mathbf{v}_c(t_0)\\ x_3(t_0) = \|\mathbf{r}(t_0)\|.\end{cases} \tag{17}$$

Proof. Under the terms of Theorem 4, the initial condition of the LTV system (16) is uniquely determined by the corresponding system output and input. The proof follows by showing that (17) explains the system output. The output of the LTV system (16) is given by

$$\mathbf{y}(t) = \mathbf{x}_1(t_0) + (t-t_0)\,\mathbf{x}_2(t_0) + \int_{t_0}^{t}\mathbf{v}_r(\tau)d\tau + x_3(t_0)\,\mathbf{d}(t) + \int_{t_0}^{t}\left[\mathbf{v}_s(\tau) - \mathbf{v}_r(\tau) - \mathbf{x}_2(t_0)\right]^T\mathbf{d}(\tau)d\tau\,\mathbf{d}(t) = \mathbf{s}(t) \tag{18}$$

for all $t \in \mathcal{I}$, $\mathcal{I} = \left[t_0, t_f\right]$. Substituting (17) in (18) gives

$$\mathbf{y}(t) = \mathbf{p}(t_0) + (t-t_0)\,\mathbf{v}_c(t_0) + \int_{t_0}^{t}\mathbf{v}_r(\tau)d\tau + \|\mathbf{r}(t_0)\|\,\mathbf{d}(t) + \int_{t_0}^{t}\left[\mathbf{v}_s(\tau) - \mathbf{v}_r(\tau) - \mathbf{v}_c(t_0)\right]^T\mathbf{d}(\tau)d\tau\,\mathbf{d}(t). \tag{19}$$

It remains only to show that (19) is equal to $\mathbf{s}(t)$ for all $t \in \mathcal{I}$. Substituting $t = t_0$ in (19) yields

$$\mathbf{y}(t_0) = \mathbf{p}(t_0) + \|\mathbf{r}(t_0)\|\,\mathbf{d}(t_0) = \mathbf{p}(t_0) + \mathbf{r}(t_0) = \mathbf{s}(t_0).$$

The time derivative of (19) is given by

$$\dot{\mathbf{y}}(t) = \|\mathbf{r}(t_0)\|\,\dot{\mathbf{d}}(t) + \mathbf{v}_r(t) + \mathbf{v}_c(t_0) + \int_{t_0}^{t}\left[\mathbf{v}_s(\tau) - \mathbf{v}_r(\tau) - \mathbf{v}_c(t_0)\right]^T\mathbf{d}(\tau)d\tau\,\dot{\mathbf{d}}(t) + \left[\mathbf{v}_s(t) - \mathbf{v}_r(t) - \mathbf{v}_c(t_0)\right]^T\mathbf{d}(t)\mathbf{d}(t). \tag{20}$$

As $\mathbf{v}_c(t)$ is constant, it is possible to rewrite (20) as

$$\dot{\mathbf{y}}(t) = \|\mathbf{r}(t_0)\|\,\dot{\mathbf{d}}(t) + \mathbf{v}_r(t) + \mathbf{v}_c(t) + \int_{t_0}^{t}\left[\mathbf{v}_s(\tau) - \mathbf{v}_r(\tau) - \mathbf{v}_c(\tau)\right]^T\mathbf{d}(\tau)d\tau\,\dot{\mathbf{d}}(t) + \left[\mathbf{v}_s(t) - \mathbf{v}_r(t) - \mathbf{v}_c(t)\right]^T\mathbf{d}(t)\mathbf{d}(t). \tag{21}$$

Using $\frac{d}{dt}\|\mathbf{r}(t)\| = [\mathbf{v}_s(t) - \mathbf{v}_r(t) - \mathbf{v}_c(t)]^T\,\mathbf{d}(t)$ one may write

$$\|\mathbf{r}(t)\| = \|\mathbf{r}(t_0)\| + \int_{t_0}^{t}\left[\mathbf{v}_s(\tau) - \mathbf{v}_r(\tau) - \mathbf{v}_c(\tau)\right]^T\mathbf{d}(\tau)d\tau. \tag{22}$$

On the other hand, the time derivative of (1) is given by

$$\dot{\mathbf{d}}(t) = \frac{\mathbf{v}_s(t) - \mathbf{v}_r(t) - \mathbf{v}_c(t)}{\|\mathbf{r}(t)\|} - \frac{\left[\mathbf{v}_s(t) - \mathbf{v}_r(t) - \mathbf{v}_c(t)\right]^T\mathbf{d}(t)}{\|\mathbf{r}(t)\|}\mathbf{d}(t). \tag{23}$$

Substituting (22) and (23) in (21) gives $\dot{\mathbf{y}}(t) = \mathbf{v}_s(t)$. This concludes the proof, as with $\mathbf{y}(t_0) = \mathbf{s}(t_0)$ and $\dot{\mathbf{y}}(t) = \mathbf{v}_s(t) = \dot{\mathbf{s}}(t)$ it must be $\mathbf{y}(t) = \mathbf{s}(t)$ for all $t \in \mathcal{I}$ and therefore (17) is true. □

The design of a Kalman filter with globally exponentially stable error dynamics for navigation based on direction measurements follows naturally as in Section 3.3.

5. Navigation filter design without the source velocity

The design for navigation aided by direction measurements presented in Section 4 requires the velocity of the source. Although that is feasible in cooperative navigation, it is also interesting to consider a scenario where $\mathbf{v}_s(t)$ is not available. This could be interesting when the source is equipped with a localization sensor but not a full navigation system. This section deals with this problem, presenting an alternative design for navigation based on direction measurements that does not require the velocity of the source.

To that purpose, notice that

$$\left[\mathbf{I}-\mathbf{d}(t)\mathbf{d}^T(t)\right]\mathbf{r}(t)=\left[\mathbf{I}-\mathbf{d}(t)\mathbf{d}^T(t)\right]\|\mathbf{r}(t)\|\,\mathbf{d}(t)=\mathbf{0}$$

for all t, which allows to write

$$\left[\mathbf{I}-\mathbf{d}(t)\mathbf{d}^T(t)\right]\mathbf{p}(t)=\left[\mathbf{I}-\mathbf{d}(t)\mathbf{d}^T(t)\right]\mathbf{s}(t). \tag{24}$$

Combining (24) with (2) gives the LTV system

$$\begin{cases}\dot{\mathbf{x}}_r(t)=\mathbf{A}_r\mathbf{x}_r(t)+\mathbf{B}_r\mathbf{u}_r(t)\\ \mathbf{y}_r(t)=\mathbf{C}_r(t)\mathbf{x}_r(t)\end{cases} \tag{25}$$

where $\mathbf{x}_r(t)=\begin{bmatrix}\mathbf{p}^T(t) & \mathbf{v}_c^T(t)\end{bmatrix}^T\in\mathbb{R}^6$ is the system state, $\mathbf{u}_r(t)=\mathbf{v}_r(t)$ is the system input,

$$\mathbf{A}_r=\begin{bmatrix}\mathbf{0} & \mathbf{I}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\in\mathbb{R}^{6\times 6},\qquad \mathbf{B}_r=\begin{bmatrix}\mathbf{I}\\ \mathbf{0}\end{bmatrix}\in\mathbb{R}^{6\times 3}$$

and $\mathbf{C}_r(t)=\begin{bmatrix}\mathbf{I}-\mathbf{d}(t)\mathbf{d}^T(t) & \mathbf{0}\end{bmatrix}\in\mathbb{R}^{3\times 6}$.

It is now important to assess about the observability of the LTV system (25) in order to apply a Kalman filter. It turns out that the results are identical to those previously derived, as detailed in the following theorems.

Theorem 7. *The LTV system (25) is observable on $\mathcal{I}:=\left[t_0,t_f\right]$ if and only if the unit vector $\mathbf{d}(t)$ is not constant on $\mathcal{I}$ or, equivalently, (4) is true.*

Proof. Let $\mathbf{c}=\begin{bmatrix}\mathbf{c}_1^T & \mathbf{c}_2^T\end{bmatrix}^T\in\mathbb{R}^6$, $\mathbf{c}_i\in\mathbb{R}^3$, $i=1,2$, be a unit vector, i.e., $\|\mathbf{c}\|=1$. Then, it is straightforward to show that

$$\mathbf{c}^T\mathcal{W}_r\left(t_0,t_f\right)\mathbf{c}=\int_{t_0}^{t_f}\|\mathbf{f}_r(\tau)\|^2\,\mathrm{d}\tau$$

for all $\|\mathbf{c}\|=1$, where $\mathcal{W}_r\left(t_0,t_f\right)$ denotes the observability Gramian associated with the pair $(\mathbf{A}_r,\mathbf{C}_r(t))$ on $\mathcal{I}$ and

$$\mathbf{f}_r(\tau)=\left[\mathbf{I}-\mathbf{d}(\tau)\mathbf{d}^T(\tau)\right]\left[\mathbf{c}_1+(\tau-t_0)\,\mathbf{c}_2\right]$$

for all $\tau\in\mathcal{I}$. The first derivative of $\mathbf{f}_r(\tau)$ is given by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{f}_r(\tau)=&-\left[\dot{\mathbf{d}}(\tau)\mathbf{d}^T(\tau)+\mathbf{d}(\tau)\dot{\mathbf{d}}^T(\tau)\right]\mathbf{c}_1+\left[\mathbf{I}-\mathbf{d}(\tau)\mathbf{d}^T(\tau)\right]\mathbf{c}_2\\&-(\tau-t_0)\left[\dot{\mathbf{d}}(\tau)\mathbf{d}^T(\tau)+\mathbf{d}(\tau)\dot{\mathbf{d}}^T(\tau)\right]\mathbf{c}_2\end{aligned}$$

for all $\tau\in\mathcal{I}$. It is easily shown that, under Assumption 1, the first two derivatives are norm-bounded, from above, on $\mathcal{I}$.

The proof of necessity follows by contraposition. Suppose that (4) is not verified. Then, the unit vector $\mathbf{d}(t)$ is constant on $\mathcal{I}$, i.e., $\mathbf{d}(t)=\mathbf{d}\left(t_0\right)$ for all $t\in\mathcal{I}$. Let $\mathbf{c}_1=\mathbf{d}\left(t_0\right)$, $\mathbf{c}_2=\mathbf{0}$. Then, $\mathbf{f}_r(\tau)=\left[\mathbf{I}-\mathbf{d}\left(t_0\right)\mathbf{d}^T\left(t_0\right)\right]\mathbf{d}\left(t_0\right)=\mathbf{0}$ for all $\tau\in\mathcal{I}$, which in turn allows to conclude that the observability Gramian $\mathcal{W}_r\left(t_0,t_f\right)$ is not invertible and the LTV system (25) is not observable on $\mathcal{I}$. Consequently, if the LTV system (25) is observable on $\mathcal{I}$, it follows that (4) is true.

To show that (4) is also a sufficient condition, suppose first that $\mathbf{c}_1\neq 0$. For $\tau=t_0$ one has $\mathbf{f}_r\left(t_0\right)=\left[\mathbf{I}-\mathbf{d}\left(t_0\right)\mathbf{d}^T\left(t_0\right)\right]\mathbf{c}_1$. If $\|\mathbf{f}_r\left(t_0\right)\|>0$, then it follows, using Proposition 1, that $\mathbf{c}^T\mathcal{W}_r\left(t_0,t_f\right)\mathbf{c}>0$. Otherwise, it must be $\mathbf{c}_1=c_1\mathbf{d}\left(t_0\right)$ for some $c_1\neq 0$. Suppose then that $\mathbf{c}_1=c_1\mathbf{d}\left(t_0\right)$. Then,

$$\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{f}_r(\tau)\right|_{\tau=t_0}=-c_1\dot{\mathbf{d}}\left(t_0\right)+\left[\mathbf{I}-\mathbf{d}\left(t_0\right)\mathbf{d}^T\left(t_0\right)\right]\mathbf{c}_2.$$

If $\left\|\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{f}_r(\tau)\right|_{\tau=t_0}\right\|>0$, it follows, using Proposition 1 twice, that $\mathbf{c}^T\mathcal{W}_r\left(t_0,t_f\right)\mathbf{c}>0$. Otherwise, if $\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{f}_r(\tau)\right|_{\tau=t_0}=\mathbf{0}$, two cases may be considered: (i) if $\dot{\mathbf{d}}\left(t_0\right)=\mathbf{0}$, it may be $\mathbf{c}_2=\mathbf{0}$ or $\mathbf{c}_2=c_2\mathbf{d}\left(t_0\right)$ for some scalar $c_2\neq 0$; or (ii) if $\dot{\mathbf{d}}\left(t_0\right)\neq\mathbf{0}$, it must be $\mathbf{c}_2=c_1\dot{\mathbf{d}}\left(t_0\right)$.

Consider first $\mathbf{c}_1=c_1\mathbf{d}\left(t_0\right)$, $\mathbf{c}_2=\mathbf{0}$. Then, using (4), it is possible to conclude that

$$\|\mathbf{f}_r\left(t_1\right)\|=\left\|c_1\left[\mathbf{I}-\mathbf{d}\left(t_1\right)\mathbf{d}^T\left(t_1\right)\right]\mathbf{d}\left(t_0\right)\right\|>0$$

and, using Proposition 1, it follows that $\mathbf{c}^T\mathcal{W}_r\left(t_0,t_f\right)\mathbf{c}>0$ for $\mathbf{c}_1=c_1\mathbf{d}\left(t_0\right)$ and $\mathbf{c}_2=\mathbf{0}$. Suppose now that $\mathbf{c}_1=c_1\mathbf{d}\left(t_0\right)$ and $\mathbf{c}_2=c_2\mathbf{d}\left(t_0\right)$. Then

$$\|\mathbf{f}_r\left(t_1\right)\|=\left|c_1+\left(t_1-t_0\right)c_2\right|\left\|\left[\mathbf{I}-\mathbf{d}\left(t_1\right)\mathbf{d}^T\left(t_1\right)\right]\mathbf{d}\left(t_0\right)\right\|.$$

If $c_1+\left(t_1-t_0\right)c_2\neq 0$, then it is possible to conclude, from (4), that $\|\mathbf{f}_r\left(t_1\right)\|>0$ and, using Proposition 1, $\mathbf{c}^T\mathcal{W}_r\left(t_0,t_f\right)\mathbf{c}>0$. Otherwise, if $c_1+\left(t_1-t_0\right)c_2=0$, there exists, by continuity, $t_0<t_2<t_1$ such that $\mathbf{d}^T\left(t_0\right)\mathbf{d}\left(t_2\right)<1$. As such

$$\|\mathbf{f}_r\left(t_2\right)\|=\left|c_1+\left(t_2-t_0\right)c_2\right|\left\|\left[\mathbf{I}-\mathbf{d}\left(t_2\right)\mathbf{d}^T\left(t_2\right)\right]\mathbf{d}\left(t_0\right)\right\|$$

is positive. Again, using Proposition 1, it follows that $\mathbf{c}^T\mathcal{W}_r\left(t_0,t_f\right)\mathbf{c}>0$ for $\mathbf{c}_1=c_1\mathbf{d}\left(t_0\right)$ and $\mathbf{c}_2=c_2\mathbf{d}\left(t_0\right)$. If $\mathbf{c}_2=c_1\dot{\mathbf{d}}\left(t_0\right)$, with $\dot{\mathbf{d}}\left(t_0\right)\neq\mathbf{0}$, there exists $\epsilon>0$ such that

$$\begin{aligned}\mathbf{f}\left(t_0+\epsilon\right)=&\,c_1\left[\mathbf{I}-\mathbf{d}\left(t_0+\epsilon\right)\mathbf{d}^T\left(t_0+\epsilon\right)\right]\mathbf{d}\left(t_0\right)\\&+\epsilon c_1\left[\mathbf{I}-\mathbf{d}\left(t_0+\epsilon\right)\mathbf{d}^T\left(t_0+\epsilon\right)\right]\dot{\mathbf{d}}\left(t_0\right)\end{aligned}$$

where $\mathbf{d}\left(t_0+\epsilon\right)$ cannot be expressed as a linear combination of $\mathbf{d}\left(t_0\right)$ and $\dot{\mathbf{d}}\left(t_0\right)$. As such, $\|\mathbf{f}\left(t_0+\epsilon\right)\|>0$ and, using Proposition 1, $\mathbf{c}^T\mathcal{W}\left(t_0,t_f\right)\mathbf{c}>0$ for $\mathbf{c}_1=c_1\mathbf{d}\left(t_0\right)$ and $\mathbf{c}_2=c_1\dot{\mathbf{d}}\left(t_0\right)$. This allows to conclude, so far, that if $\mathbf{c}_1\neq\mathbf{0}$, then $\mathbf{c}^T\mathcal{W}\left(t_0,t_f\right)\mathbf{c}>0$. Suppose now that $\mathbf{c}_1=\mathbf{0}$, which implies that $\|\mathbf{c}_2\|=1$. Then, $\mathbf{f}_r\left(t_0\right)=\mathbf{0}$ and

$$\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{f}_r(\tau)\right|_{\tau=t_0}=\left[\mathbf{I}-\mathbf{d}\left(t_0\right)\mathbf{d}^T\left(t_0\right)\right]\mathbf{c}_2.$$

If $\mathbf{c}_2\neq\pm\mathbf{d}\left(t_0\right)$, then

$$\left\|\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{f}_r(\tau)\right|_{\tau=t_0}\right\|>0$$

and using Proposition 1 twice, it is possible to conclude that $\mathbf{c}^T\mathcal{W}\left(t_0,t_f\right)\mathbf{c}>0$. Otherwise, if $\mathbf{c}_2=\pm\mathbf{d}\left(t_0\right)$ then

$$\|\mathbf{f}_r\left(t_1\right)\|=\left|t_1-t_0\right|\left\|\left[\mathbf{I}-\mathbf{d}\left(t_1\right)\mathbf{d}^T\left(t_1\right)\right]\mathbf{d}\left(t_0\right)\right\|>0.$$

Again, using Proposition 1, it is possible to conclude that $\mathbf{c}^T\mathcal{W}\left(t_0,t_f\right)\mathbf{c}>0$ for $\mathbf{c}_1=\mathbf{0}$ and $\mathbf{c}_2=\pm\mathbf{d}\left(t_0\right)$. But that concludes the proof, as it was shown that $\mathbf{c}^T\mathcal{W}\left(t_0,t_f\right)\mathbf{c}>0$ for all $\|\mathbf{c}\|=1$, which means that (25) is observable. □

Theorem 8. *The LTV system (25) is uniformly completely observable if and only if*

$$\exists_{\substack{\alpha>0\\ \delta>0}}\ \forall_{t\geq t_0}\ \int_t^{t+\delta}\left[\mathbf{d}^T(t)\,\mathbf{d}(\tau)\right]^2\mathrm{d}\tau\leq\delta\,(1-\alpha). \tag{26}$$

Proof. The proof of sufficiency follows similar steps to Theorem 7 considering uniformity bounds that stem from the persistent excitation condition (26). Therefore it is omitted. To show that (26) is also necessary, suppose that (26) does not hold. Then,

$$\forall_{\substack{\alpha>0\\ \delta>0}}\ \exists_{t^*\geq t_0}\ \int_{t^*}^{t^*+\delta}\left[\mathbf{d}^T\left(t^*\right)\mathbf{d}(\tau)\,\mathrm{d}\tau\right]^2>\delta\,(1-\alpha). \tag{27}$$

Let $\mathbf{c}=\begin{bmatrix}\mathbf{d}^T\left(t^*\right) & \mathbf{0}\end{bmatrix}^T\in\mathbb{R}^6$. Then, it is easily shown that

$$\mathbf{c}^T\mathcal{W}_r\left(t^*,t^*+\delta\right)\mathbf{c}=\int_{t^*}^{t^*+\delta}\left(1-\left[\mathbf{d}\left(t^*\right)\mathbf{d}(\tau)\right]^2\right)\mathrm{d}\tau. \tag{28}$$

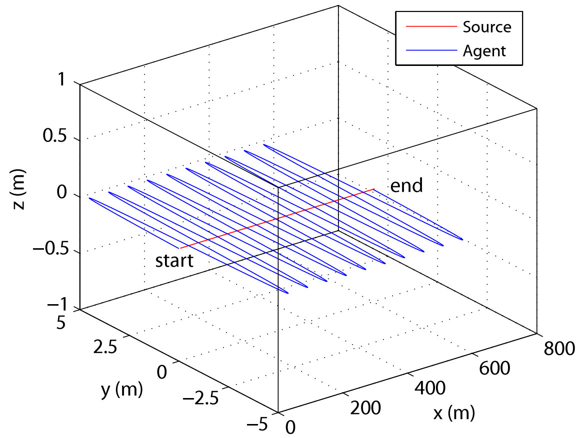

Fig. 1. Trajectory described by the agent and the source.

Using (27) in (28) allows to conclude that, for all $\alpha > 0$ and $\delta > 0$ there exists a time instant $t^* \geq t_0$ such that $\mathbf{c}^T \mathcal{W}(t^*, t^* + \delta)\,\mathbf{c} < \delta\alpha$, which means that the LTV system (25) is not uniformly completely observable. Therefore, if the LTV system (25) is uniformly completely observable, (25) is true. □

Remark 2. Notice that (26) is true if and only if (12) is true. The former was preferred in this section because it simplifies the proof of Theorem 8.

6. Simulation results

This section presents realistic simulation results for the source localization problem in order to evaluate the performance achieved with the proposed solutions. Further testing revealed that similar performances are achieved for the navigation problem based on direction measurements.

In the simulations, the source and the agent trajectories are those depicted in Fig. 1. Clearly, the persistent excitation condition (12) is satisfied, which allows to apply the solutions proposed in the paper. The drift velocity of the source was set to $\mathbf{v}_s(t) = [1\ 0\ 0]^T$ (m/s), while the drift velocity of the agent was set to $\mathbf{v}_c(t) = [-0.5\ 0\ 0]^T$ (m/s), which gives $\mathbf{v}_{sa}(t)\,[1.5\ 0\ 0]^T$ (m/s) for the relative drift velocity.

Noise was considered for both the direction measurements and the relative velocity of the agent $\mathbf{v}_r(t)$. In particular, additive zero mean white Gaussian noise was considered for $\mathbf{v}_r(t)$, with standard deviation of 0.01 m/s, while the direction readings were assumed perturbed by rotations about random vectors of an angle modeled by zero-mean white Gaussian noise, with standard deviation of 1°. The Kalman filter parameters were set to $\boldsymbol{\Xi} = \operatorname{diag}\left(10^{-2}\mathbf{I}, 10^{-5}\mathbf{I}, 10^{-2}\right)$ and $\boldsymbol{\Theta} = \mathbf{I}$. The initial estimates were all set to zero.

The evolution of the estimation error is shown in Fig. 2. As it is possible to see, the initial transients due to the mismatch of the initial conditions quickly fade out, resulting in state estimates very close to the true value.

In order to better evaluate the performance of the proposed solution, the Monte Carlo method was applied. The simulation was carried out 1000 times with different, randomly generated noise signals. The mean and standard deviation were computed for each simulation and averaged over the 1000 simulations. The results are depicted in Table 1, where the results obtained with an Extended Kalman Filter with similar parameters are also presented. As the initial estimate for the source location cannot be set to zero with the EKF (in the linearization there appear terms divided by the norm of this estimate), the initial source position estimate was set to [1 0 0] m. The convergence speed results slightly smaller. As it is possible to observe, the proposed solutions achieve similar performance to the EKF, while providing, at the same time, global exponential stability guarantees.

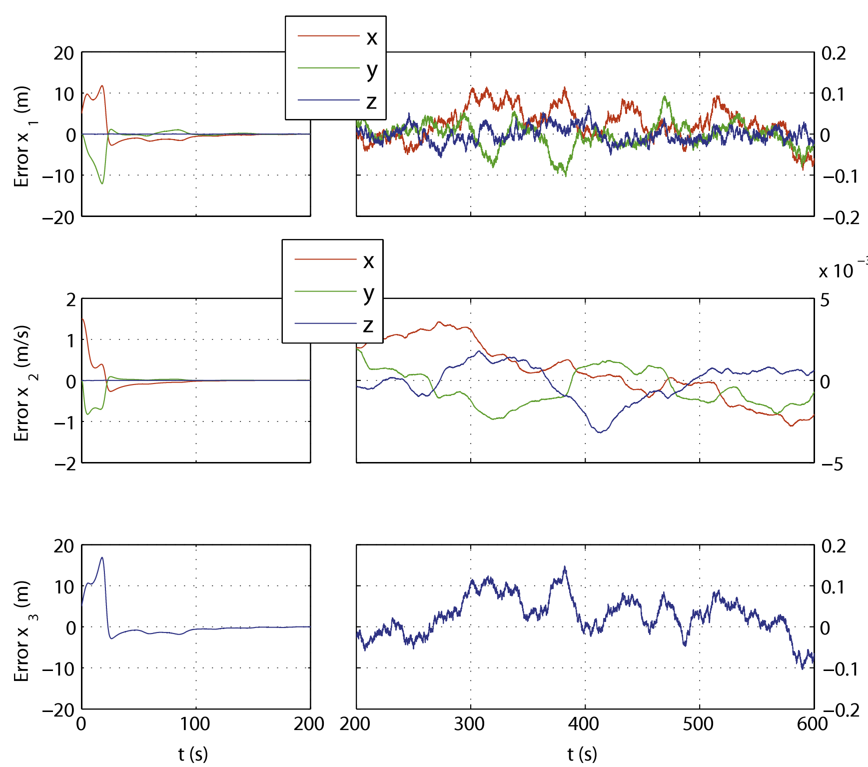

Fig. 2. Evolution of the estimation errors.

Table 1
Standard deviation of the steady-state estimation error, averaged over 1000 runs of the simulation.

	Proposed solution	EKF
$\sigma_{\tilde{x}_{11}}$ (m)	8.5×10^{-3}	19.6×10^{-3}
$\sigma_{\tilde{x}_{12}}$ (m)	3.3×10^{-3}	11.7×10^{-3}
$\sigma_{\tilde{x}_{13}}$ (m)	1.2×10^{-3}	9.2×10^{-3}
$\sigma_{\tilde{x}_{21}}$ (m/s)	4.8×10^{-4}	5.6×10^{-4}
$\sigma_{\tilde{x}_{22}}$ (m/s)	4.8×10^{-4}	5.8×10^{-4}
$\sigma_{\tilde{x}_{23}}$ (m/s)	4.6×10^{-4}	5.8×10^{-4}
$\sigma_{\tilde{x}_{3}}$ (m)	1.1×10^{-2}	Not explicitly estimated

7. Conclusions

This paper presented a set of globally exponentially stable Kalman filters for the problems of source localization and navigation based on direction measurements to a single source. The observability of the systems was fully characterized, which allowed to conclude about the asymptotic stability of the Kalman filters. The observability conditions that were derived are directly related to the motion of the agent/vehicle and as such they are useful for motion planning and control. Simulation results were presented that illustrate the good performance achieved by the proposed solutions, which were also compared with the EKF, achieving similar performance but with global asymptotic stability guarantees. Future work includes the extension of the present work to the case where directions to multiple sources are available for navigation purposes.

Acknowledgments

This work was partially supported by the FCT, reference [PEst-OE/EEI/LA0009/2013], by the project MAST/AM of the FCT, reference PTDC/EEA-CRO/111197/2009, and by the EU project TRIDENT of the EC-FP7.

References

[1] B. Fidan, S. Dandach, S. Dasgupta, B. Anderson, A continuous time linear adaptive source localization algorithm robust to persistent drift, Systems & Control Letters 58 (1) (2009) 7–16.

[2] P. Batista, C. Silvestre, P. Oliveira, Single range aided navigation and source localization: observability and filter design, Systems & Control Letters 60 (8) (2011) 665–673.

[3] M. Deghat, I. Shames, B. Anderson, C. Yu, Target localization and circumnavigation using bearing measurements in 2D, in: Proceedings of the 49th IEEE Conference on Decision and Control, Atlanta, GA, USA, 2010, pp. 334–339.

[4] S. Nardone, V. Aidala, Observability criteria for bearings-only target motion analysis, IEEE Transactions on Aerospace and Electronic Systems 17 (2) (1981) 162–166.

[5] S. Hammel, V. Aidala, Observability requirements for three-dimensional tracking via angle measurements, IEEE Transactions on Aerospace and Electronic Systems 21 (2) (1985) 200–207.

[6] K. Becker, A general approach to TMA observability from angle and frequency measurements, IEEE Transactions on Aerospace and Electronic Systems 32 (1) (1996) 487–494.

[7] K. Becker, Three-dimensional target motion analysis using angle and frequency measurements, IEEE Transactions on Aerospace and Electronic Systems 41 (1) (2005) 284–301.

[8] P. Batista, C. Silvestre, P. Oliveira, Single beacon navigation: observability analysis and filter design, in: Proceedings of the 2010 American Control Conference, Baltimore, USA, 2010, pp. 6191–6196.

[9] D. Viegas, P. Batista, P. Oliveira, C. Silvestre, Linear motion observers for ASC/AUV tandems based on single range readings, in: Proceedings of the 2011 American Control Conference, San Francisco, USA, 2011, pp. 3966–3971.

[10] A. Gadre, D. Stilwell, A complete solution to underwater navigation in the presence of unknown currents based on range measurements from a single location, in: Proceedings of the 2005 IEEE/RSJ International Conference on Intelligent Robots and Systems, Edmonton AB, Canada, 2005, pp. 1420–1425.

[11] T. Casey, B. Guimond, J. Hu, Underwater vehicle positioning based on time of arrival measurements from a single beacon, in: Proceedings of the MTS/IEEE Oceans 2007, Vancouver, BC, Canada, 2007, pp. 1–8.

[12] P.-M. Lee, B.-H. Jun, K. Kim, J. Lee, T. Aoki, T. Hyakudome, Simulation of an inertial acoustic navigation system with range aiding for an autonomous underwater vehicle, IEEE Journal of Oceanic Engineering 32 (2) (2007) 327–345.

[13] K. Briechle, U. Hanebeck, Localization of a mobile robot using relative bearing measurements, IEEE Transactions on Robotics and Automation 20 (1) (2004) 36–44.

[14] P. Batista, C. Silvestre, P. Oliveira, Globally asymptotically stable filters for source localization and navigation aided by direction measurements, in: Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference, Orlando, Florida, USA, 2011.

[15] P. Batista, C. Silvestre, P. Oliveira, On the observability of linear motion quantities in navigation systems, Systems & Control Letters 60 (2) (2011) 101–110.

[16] W. Rugh, Linear System Theory, second ed., Prentice-Hall, Inc., 1995.

[17] A. Jazwinski, Stochastic Processes and Filtering Theory, Academic Press, Inc., 1970.

[18] R.W. Brockett, Finite Dimensional Linear Systems, Wiley, 1970.