Stability of re-entrant flow lines*

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Abstract The paper addresses the issue of stability for re-entrant flow lines, producing multiple products, with capacitated machines, random demand and random yield. The policies of interest are multi-echelon base stock policies, combined with a set of static and dynamic management rules of the available capacity. We introduce three classes of capacitated echelon base stock policies: Π_0, the pure multi-echelon base stock; Π_1, like Π_0 with a possibly finite upper bound on the admission of raw materials; and Π_2, like Π_1 with a possibly finite upper bound on the utilization of intermediate inventories. The order of business is: establishing conditions for the stability of the shortfall echelon process when demands are stationary and ergodic; examining the regenerative structure of the shortfall process when demands are given by an i.i.d. sequence. The regenerative properties are valuable in establishing the convergence of costs and also simulation estimators, which enables the utilization of Infinitesimal Perturbation Analysis to optimize the policy parameters. We use a coupling argument for shortfalls while establishing the stability conditions, which will, by itself, render the Harris ergodicity of the shortfall process. We show that the stability condition suffices to ensure that the shortfall process possesses the regenerative structure of a Harris ergodic Markov chain. Under a stronger condition, we establish that the vector of shortfalls returns to the origin infinitely often, with probability one. We show that the necessary stability condition is also sufficient for any sort of re-entrant system, in the presence of random yield, provided the control policy is in Π_2.

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1 Introduction

A framework to manage re-entrant flow lines producing multiple products was proposed in [6]. The work focused the analysis on a simple set of capacity managing schemes and production rules as a first step towards understanding broader classes of systems and policies. The re-entrant lines were modeled as discrete time capacitated multi-product production/inventory systems, subject to random demand, operating under multi-echelon base stock policies. Several capacity sharing mechanisms were discussed and some production rules were proposed to both statically and dynamically manage capacity.

An Infinitesimal Perturbation Analysis (IPA) approach was proposed and validated, in order to compute the optimal values of the parameters describing the control policies. In order to validate the infinite horizon measures and derivatives, one has to rigorously establish the stability conditions for the systems being addressed. The main objective of this paper is exclusively address this problem for re-entrant flow lines.

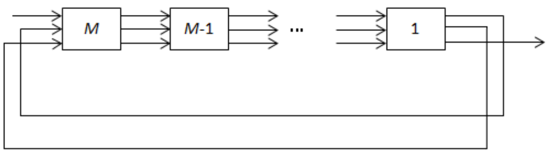

Fig. 1 Schematic of the default re-entrant system.

The first set of systems under consideration has a series of M machines (stages), and each of the P products has to cycle K times (levels) through each of the M stages. At any given period, each machine may process different parts belonging to different levels (see Figure 1). The total production per period is limited by the machine capacity and feeding inventory. After being processed by a machine, parts are placed in intermediate buffers where they wait their turn to be processed by the next machine or until they are depleted by external demand.

Each level and stage operates on a base stock policy for echelon inventory. The capacity is managed both from a static and a dynamic approach. In what concerns the static capacity management, the capacity of each machine, C^m, with $m = 1,2,\dots M$, may be split into $K \times P$ slots, C^{kmp}, with $k = 1,2,\dots K$ and $p = 1,2,\dots P$, each assigned to a specific level/product pair (NS – no sharing). Alternatively, the capacity may be split into K slots, C^{km}, each assigned to a specific level and shared by all products at that level (PS – partial sharing). Another possibility is to consider the available capacity as being simultaneously shared by all products and levels (TS – total sharing). As to the dynamic capacity management, whenever there is some degree of capacity sharing, a set of production rules may be defined. Examples are Linear Scaling, Priority, and Equalize Shortfall.

1.1 Literature Review

We start by reviewing the relevant literature on stability for production systems. The review covers three main modeling paradigms: differential equations, queuing networks, and difference equations. These are among the main tools used to model production systems and all of them have provided contributions and relevant insights to this of paper.

When approaching the problem of production control by means of formulating an optimal control problem, stability questions are trivially answered in the following sense: if there is one stable policy, the optimal policy will also be stable, provided the performance measure is adequate, and it will be found through the optimization procedure. However, when the approach is to choose a class of policies a priori, stability has to be addressed explicitly, in order to determine whether or not the given class ensures it.

In the optimal control formulation of [19], there is no explicit consideration of stability. Demand for each product is assumed to be a deterministic constant rate, processing times are deterministic, and machines are subject to random failures. The inventory dynamics are described by differential equations and the failure process is modeled through a Markov process. Even though the approach is applied to re-entrant systems in [2] or to generic job shops [12], the necessary stability condition is assumed to be sufficient. In fact, stability was never addressed explicitly by the authors who have done work on flow rate control, except for [7]. The authors explicitly state and prove that as long as the demand vector is an interior point of the expected capacity set, then there exists a flow control policy that results in a stable system. Stability is taken in the sense that the expected end product inventory is finite for all products. The proof does not rely on any specific assumption on the flow patterns inside the production system and it only accounts for end product inventories.

When modeling production systems by means of queueing networks and proposing specific scheduling rules, it often has been the case that stability becomes a hard question to answer. Also, to establish the heavy traffic limit theorems of [15, 16], it is necessary to establish the stability of the queueing networks considered. Examples of networks for which the Brownian approximation does not hold have been presented, as [10] is one example. Usually, the issue of stability in networks of queues is established by explicitly determining an invariant distribution. The classes of queueing networks for which such invariant distribution is known are very limited. Typically, networks of queues operated under local scheduling policies are among those for which little is known about their invariant distribution or even if one exists. They fall outside the classes for which there are product form solutions. Product form solutions exist for the generalized Jackson networks: single class networks with exponential inter arrival and service times, where queues are served in a First come First serve order, [17]. For some scheduling disciplines in multiclass networks, with special distributional assumptions on inter arrival and service times, the stationary distributions were explicitly determined in [4, 18]. One example of open queueing networks where addressing the stability problem is highly relevant for this paper is the work of [23]. Their distributed CAF policies for local scheduling are ensured to be stable for acyclic systems as long as the necessary stability condition holds – traffic intensity being less than one for all servers. However, the authors were unable to show similar properties for non-acyclic systems and had to propose a modification of the original policies to stabilize any such system. One of the central statements of the present paper is that the structure of the modification proposed holds the key to the problem of stability. Also, this seems to have been overlooked by other authors. Their original distributed CAF policies are non-idling, or work-conserving as some authors prefer to call them. The problem addressed in [23] was one of reducing the number of set-ups as much as possible, since a set-up is a waste of capacity. Nevertheless, when the authors found themselves unable to prove their policies to be stable for non-acyclic systems, they proposed a modification that basically increases the number of set-ups, thus incurring more waste of capacity. Besides this, they also allowed each server to remain idle, when not in a set-up – distributed CAF policies with backoff –, even if there would be other jobs in the queue. Idling policies are simpler to model in the context of flow rate control, as the two boundary policy of [24] is an example. For a system with two machines in tandem, when the production trails demand by a large margin, the policy imposes a bound on the amount of inventory between machine 1 and machine 2, thus preventing the first machine to work at its maximum rate. In the context of queueing networks, similar ideas could be applied. The work of [22] is a contribution along these lines. However, this type of ideas has not been pursued as intensively as one would expect, taking into account what is known from the deterministic scheduling theory, [3, 11]. One recent example where idling policies are explicitly used is [9].

In [23], the inability to prove stability for the original policies could be thought of as a problem that would be solved in due time, since the authors did not show that in fact instability could occur. This question ended up being answered through an example not much later. In [20], also for open queueing systems, the authors introduce an example of a re-entrant system for which there exists a non-idling control policy that yields unbounded trajectories for the buffer sizes, although the workload imposed by demand is below the available capacity. In the recent years many other such examples were presented. For the sake of brevity we refer the reader to the review presented in [8], which constitutes a good synthesis of the research on queuing systems, concerning the particular issue of stability.

For our purposes, suffices to say that the great majority of the efforts in the area of modeling production systems through networks of queues has concentrated on idling policies and on determining sufficient conditions for stability which are more restrictive than the traffic condition.

In [14], the validation of the Infinitesimal Perturbation Analysis approach for infinite horizon costs relies on the proof of stability for the single-product, multiple-machine, and non re-entrant system presented in [13]. They model the production systems by means of difference equations, with deterministic capacity and random demand. The authors show that it suffices to have the expected demand below the capacity of the machine with the lowest output in order to ensure their control policies to be stable. Although dealing with a non re-entrant system, for which the stability issue is trivial, the discussion on stability is useful and necessary to identify renewal points of the stochastic processes considered, which has implications on the validation of the approach to estimate values and gradients of infinite horizon performance measures.

From the perspective of the present paper, the concern regarding stability is twofold: one the one hand, there is a need similar to that of [13] regarding the validation of the Infinitesimal Perturbation Analysis to bring formal closure to [6]; on the other hand, it is necessary to verify if the control policies proposed at that time ensure stability, given the fact that they are applied to systems with more complex flow patterns.

We also have the purpose of contributing to the stability discussion for non-acyclic systems. In the remaining of this paper, we show that it is necessary to depart from non idling policies. Given the modeling paradigm adopted, it will be easy to define a class of policies containing both idling and non-idling policies as sub-classes.

1.2 Brief model review

We review the essentials of the base model. We refer interested readers to the more extensive discussion in [6]. Let the echelon inventory of product p at machine m and level k for period n be given by

$$E_n^{kmp} = I_n^{kmp} + E_n^{(km)^- p}, \tag{1}$$

where $(km)^-$ designates the level and stage that fed by level k at stage m; I_n^{kmp} is the amount of inventory of product p, level k, and stage m; and $E_n(11)^- p = 0$ for all p.

The echelon shortfall is given by

$$Y_n^{kmp} = z^{kmp} - E_n^{kmp}, \tag{2}$$

where z^{kmp} are the echelon base stock parameters. The dynamic equations for the echelon shortfall are

$$Y_{n+1}^{kmp} = Y_n^{kmp} + d_n^p - P_n^{kmp}, \tag{3}$$

with d_n^p denoting the external demand for the end product p on period n and P_n^{kmp} denoting the production decision. An instance of the production decision, for a system being operated with PS and the Linear Scaling rule, would be

$$P_n^{kmp} = f_n^{kmp} g_n^{km} \tag{4}$$

where f_n^{kmp}, the net production request of product p, level k for stage m, is given by

$$f_n^{kmp} = \min\left\{(z^{kmp} + d_n^p - E_n^{kmp})^+, I_n^{(km)^+ p}\right\}, \tag{5}$$

with $(x)^+ = \max\{0, x\}$ and $(km)^+$ the level and stage which feeds I_n^{kmp}, with the raw material $I_n^{(KM)^+ p} = \infty$. The term g_n^{km} is given by

$$g_n^{km} = \min\left\{ \frac{C^{km}}{\sum_p f_n^{kmp}}, 1 \right\}, \tag{6}$$

where C^{km} is the slot of C^m assigned to level k. The expression above assumes that each product, irrespective of the level, will require the same amount of machine capacity per unit – uniform load assumption. The non-uniform load situation takes as τ^{kmp} the capacity needs of a single unit of product p, on machine m and level k. Therefore, the above expression would be written as

$$g_n^{km} = \min\left\{ \frac{C^{km}}{\sum_p \tau^{kmp} f_n^{kmp}}, 1 \right\}. \tag{7}$$

In this paper we generalize this model to include different machine loads for each product and level, the presence of random yield, and consider that the re-entrant flow may have no restrictions with alternative routs of different lengths for each of the products.

We now introduce three classes of policies.

Definition 1. *Capacitated, Echelon Base Stock Policy:* Let Y_n^{kmp} be the shortfall at period n for product p, level k, and stage m, with $p = 1, 2, \ldots P$, $k = 1, 2, \ldots, K$, and $m = 1, 2, \ldots, M$. We define as Π_0 the class of policies that use equation (5) to determine the net production requests and use some combination of static and dynamic capacity allocation to distribute capacity among all the requests.

Definition 2. Π_0 *with external bound:* Under the conditions of Definition 1, we define as Π_1 the set of policies where there is a, possibly finite and constant, input bound of raw material. That is, $I_n^{(KM)^+ p} = I_b^p \leq \infty$ for all $p = 1, 2, \ldots, P$.

Definition 3. Π_1 *with internal bounds:* Define as Π_2 the class of policies that act like Π_1 with the addition of bounds on the amount of material allowed to enter production for each product at each machine on any given period and level. That is, the policies for which $I_n^{(km)^+ p} = I_b^{(km)^+ p} \leq \infty$.

It should be self-evident that $\Pi_0 \subset \Pi_1 \subset \Pi_2$. It should also be clear that all policies in class Π_0 are non-idling, whereas the other two classes contain idling policies. The values I_b^p and $I_b^{(km)^+ p}$ are user defined parameters, as are the parameters z^{kmp}, and therefore also subject to optimization when looking for the best performance.

The order of business is: establishing conditions for the stability of the shortfall echelon process when demands are stationary and ergodic; examining the regenerative structure of $\{Y_n, n \geq 0\}$ when $\{D_n = [d_n^1, d_n^2, \ldots, d_n^P]', n \geq 0\}$ is an i.i.d. sequence. The regenerative properties are valuable in establishing convergence of costs and also simulation estimators.

We show that the stability condition suffices to ensure that $\{Y_n, n \geq 0\}$ possesses the regenerative structure of a *Harris ergodic* Markov chain. Under a stronger condition, we establish that the vector of shortfalls returns to the origin infinitely often, with probability one.

A powerful tool in the analysis of Harris ergodic Markov chains is a connection with coupling. The main result is this: a Markov chain with an invariant probability measure admits coupling if and only if it is Harris ergodic. We use a coupling argument for shortfalls while establishing the stability conditions, which will, by itself, render the Harris ergodicity of the shortfall process.

2 Harris recurrence and explicit regeneration points

An extensive coverage of key definitions and results of this framework can be found in [1] and [21]; the treatment in [25] is particularly relevant to this application. We review the essentials here. The general setting for Harris recurrence is a Markov chain $X = \{X_n, n \geq 0\}$ on a state space $\mathbf{S}$ with Borel sets $\mathscr{B}$. Let P_x denote the law of X when $X_0 = x$. Then, X is Harris recurrent if there exists a σ-finite measure on ($\mathbf{S}$, $\mathscr{B}$), not identically zero, such that, for all $A \in \mathscr{B}$,

$$\psi(A) > 0 \Rightarrow P_x(\sum_{n=0}^{\infty} 1\{X_n \in A\} = \infty) = 1 \text{ for all } x \in \mathbf{S}.$$

Thus, every set of positive ψ-measure is visited infinitely often from all initial states. Every Harris recurrent Markov chain has an invariant measure π that is unique up to a multiplication by a constant. The sets of positive π-measure are precisely those that are visited infinitely often from all initial states. If π is finite (hence a probability, without loss of generality), then X is called *positive* Harris recurrent. If, in addition, X is aperiodic, then it is Harris ergodic.

The connection with regeneration enters as follows. If X is Harris recurrent, then there exists a (discrete-time) renewal process $\{\tau_k, k \geq 1\}$ and an integer $r \geq 1$ such that

$$\{(X_{k+n}, n \geq 0), (\tau_{n+k+1} - \tau_{n+k} n \geq 0)\},$$

has the same distribution for all $k \geq 1$ and is independent of

$$\{\tau_1, \ldots, \tau_k, (X_n, 0 \leq n \leq \tau_k - r)\}.$$

When $r > 1$, there may be dependence between consecutive cycles $\{X_n, \tau_{k-1} \leq n \leq \tau_k\}$, in contrast to the classical case of independent cycles (and this is indeed the case in this model). However, if X is positive Harris recurrent and if $f : \mathrm{S} \to \Re$ is π-integrable, then the regenerative ratio formula

$$\mathbf{E}_\pi[f(X_0)] = \frac{\mathbf{E}[\sum_{n=\tau_{k-1}}^{\tau_k - 1} f(X_n)]}{\mathbf{E}[\tau_k - \tau_{k-1}]},$$

remains valid, as does the associated central limit theorem (under second-moment assumptions). Moreover, if X is Harris ergodic then for all initial conditions the distribution of X_n converges to π in *total variation*, that is,

$$\sup_{A \in \mathscr{B}} |P_x(X_n \in A) - \pi(A)| \to 0,$$

as $n \to \infty$, for all $x \in \mathbf{S}$. Indeed, this total variation convergence to a probability measure completely characterizes Harris ergodicity. A powerful tool in the analysis of Harris ergodic Markov chains is a connection with coupling; see for example [26] and [25] for background. The main result is this: a Markov chain with an invariant probability measure admits coupling if and only if it is Harris ergodic. We use a coupling argument for shortfalls while establishing the stability conditions, which will by itself render the Harris ergodicity of the shortfall process.

While Harris recurrence ensures the existence of (wide-sense) regeneration times $\tau_k, k \geq 1$, it does not provide a means of identifying these times. Explicit regeneration times are not needed for convergence results, but they are useful in, for example, computing confidence intervals from simulation estimators. At the end of each section we give a sufficient condition for $\{Y_n, n \geq 0\}$ to have readily identifiable regeneration times.

In what follows, we will establish stability for all sharing modes, multiple products, in the presence of random yield, and non-uniform loads. The no sharing (NS) case has been established in [13] for perfect yield and uniform loads. The detailed analysis will be made for the case with uniform loads and perfect yield, as that constitutes the formal closure of [6]. Then, the random yield case with uniform loads will be established. This later is sufficiently general on one hand and allows an easy generalization for the case of non-uniform loads. The stability conditions for the other cases will be simply stated without proof at the end of the paper.

3 Uniform loads and perfect yield

There are two settings for which to investigate the stability conditions: when capacity is partially shared (PS); and when capacity is totally shared (TS). The no sharing (NS) case has been established in [13]. Under adequate changes, the same technique will be used to prove stability for the PS mode. Then, using a stochastic dominance argument, the conditions under which the TS case is stable will be established. We assume that the production decisions are taken with the use of the Linear Scaling Rule. It is a trivial exercise to show that all other shortfall based production rules will produce the same stability results.

3.1 The partial sharing mode

For the PS mode, we first define a dynamic equation for a linear combination of shortfalls such that the theoretical framework of [13] is readily applicable. The dynamic equation for the shortfall quantities is given by (4). To simplify the analysis we can define a vectorial dynamic equation for each stage and level by defining $\boldsymbol{D}_n = [d_n^1\ d_n^2\ \dots\ d_n^P]^T$, $\boldsymbol{P}_n^{km} = [P_n^{km1}\ P_n^{km2}\ \dots\ P_n^{kmP}]^T$, and $\boldsymbol{Y}_n^{km} = [Y_n^{km1}\ Y_n^{km2}\ \dots\ Y_n^{kmP}]^T$. Therefore, the vectorial dynamic equation will assume the form

$$\boldsymbol{Y}_{n+1}^{km} = \boldsymbol{Y}_n^{km} + \boldsymbol{D}_n - \boldsymbol{P}_n^{km} \quad \text{for all } k \text{ and } m.$$

Let $||x||$ be defined as the sum of all components of x. $||x||$ is not a norm and it verifies the following

$$\begin{aligned} ||x+y|| &= ||x|| + ||y|| \\ ||ax|| &= a||x|| \end{aligned} \tag{8}$$

Now, since level and stage (K,M) draws raw material from an infinite source, we have

$$\boldsymbol{Y}_{n+1}^{KM} = \max\{\boldsymbol{0}, (\boldsymbol{Y}_n^{KM} + \boldsymbol{D}_n)(1 - \frac{C^{KM}}{||\boldsymbol{Y}_n^{KM} + \boldsymbol{D}_n||})\} \tag{9}$$

Due to the structure of the above the following operation is valid

$$||\boldsymbol{Y}_{n+1}^{KM}|| = \max\{0, ||\boldsymbol{Y}_n^{KM}|| + ||\boldsymbol{D}_n|| - C^{KM}\} \tag{10}$$

which is a Lindley equation. Note the use of (8).

For the remaining cases we will have

$$\begin{aligned} \boldsymbol{Y}_{n+1}^{km} = \max\{&\boldsymbol{Y}_n^{km} + \boldsymbol{D}_n - \frac{C^{km}\min\{\boldsymbol{Y}_n^{km} + \boldsymbol{D}_n, \boldsymbol{I}_n^{(km)^+}\}}{||\min\{\boldsymbol{Y}_n^{km} + \boldsymbol{D}_n, \boldsymbol{I}_n^{(km)^+}\}||}, \\ &\boldsymbol{0}, \boldsymbol{Y}_n^{km} + \boldsymbol{D}_n - \boldsymbol{I}_n^{(km)^+}\} \end{aligned} \tag{11}$$

From this equation it is possible to compute $||\boldsymbol{Y}_{n+1}^{km}||$ as follows

$$||\boldsymbol{Y}_{n+1}^{km}|| = \max\Big\{0, ||\boldsymbol{Y}_n^{km}|| + ||\boldsymbol{D}_n|| - C^{km},$$
$$\sum_{p=1}^{P}\Big(Y_n^{(km)^+p} + d_n^p - \big(z^{(km)^+p} - z^{kmp}\big)\Big)^+\Big\}. \tag{12}$$

The scalar equations (10) and (12) are the multiple product generalizations of the dynamic equations for shortfalls presented in [13] for single product systems.

3.1.1 The Stationary Regime

Let us now introduce the framework and notation corresponding to Lemmas 1 and 2 of [13] which help establishing the stability conditions.

Lemma 1. The echelon shortfalls satisfy $\boldsymbol{Y}_{n+1} = \boldsymbol{\Phi}(\boldsymbol{Y}_n, \boldsymbol{D}_n)$ where $\boldsymbol{\Phi} : R_+^{KMP} \times R^P \to R_+^{KMP}$ is defined by (9, 11). Also, the total shortfall satisfy $||\boldsymbol{Y}_{n+1}|| = \phi(\boldsymbol{Y}_n, \boldsymbol{D}_n) = ||\boldsymbol{\Phi}(\boldsymbol{Y}_n, \boldsymbol{D}_n)||$ where $\phi : R_+^{KMP} \times R^P \to R_+^{KM}$ is defined by (10, 12). In particular, ϕ is increasing and continuous.

Supposing that the demands form a stationary process, without loss of generality, we can assume that $\boldsymbol{D}_n$ is defined for all integer n with $\{\boldsymbol{D}_n, -\infty < n < \infty\}$ stationary. In what follows We will use $\Rightarrow$ to denote convergence in distribution. Some of the proofs will be omitted here to avoid excessive clutter. Some of them are relatively trivial extensions of similar results published. Some others are exactly the same. Some of the former will be presented in Appendix 7 for the sake of completeness of the present document.

Lemma 2. Let $\{\boldsymbol{D}_n, -\infty < n < \infty\}$ be stationary. There exists a (possibly infinite) stationary process $\{\tilde{\boldsymbol{Y}}_n, -\infty < n < \infty\}$ satisfying $||\tilde{\boldsymbol{Y}}_{n+1}|| = \phi(\tilde{\boldsymbol{Y}}_n, \boldsymbol{D}_n)$ for all n, such that if $||\boldsymbol{Y}_0|| = 0$, a.s., then $||\boldsymbol{Y}_n|| \Rightarrow ||\tilde{\boldsymbol{Y}}_0||$.

Proof: See Appendix 7.

With the support of the above two Lemmas it is now easy to establish the stability condition for this model.

Theorem 1. Suppose the demands $\{\boldsymbol{D}_n, -\infty < n < \infty\}$ are ergodic as well as stationary. If

$$\mathbf{E}[||\boldsymbol{D}_0||] = \sum_{p=1}^{P} \mathbf{E}[d_0^p] < \min\{C^{km} : k = 1, \ldots, K; m = 1, \ldots, M\}, \tag{13}$$

then $||\tilde{\boldsymbol{Y}}_0||$ is almost surely finite. If for some (k,m), $\mathbf{E}[||\boldsymbol{D}_0||] > C^{km}$, then $||\tilde{\boldsymbol{Y}}_0^{qr}|| = \infty$, a.s., for all (q,r) corresponding to levels and stages coming after (k,m).

Proof: See Appendix 7.

This result for the scalar dynamic equations implies the stability of the vectorial process.

Corollary 1. Under the assumptions of Theorem 1, $\tilde{Y}_0^{kmp}$ is almost surely finite for all p, where $\tilde{Y}_0^{kmp}$ denotes component p of $\tilde{\boldsymbol{Y}}_0^{km}$.

Proof: The result follows trivially due to the non negativity of the shortfalls.

QED

The above results show that the process $\{\boldsymbol{Y}_n, n \geq 0\}$ converges to a stationary distribution only if $\boldsymbol{Y}_0 = 0$. The following theorem establishes that the convergence occurs for any initial point, that is, the process admits coupling.

Theorem 2. Under the stability condition $\mathbf{E}[||\boldsymbol{D}_0||] < \min_{k,m}\{C^{km}\}$, the echelon shortfall process admits coupling. Consequently, its stationary distribution is unique, and $\boldsymbol{Y}_n \Rightarrow \tilde{\boldsymbol{Y}}_0$ for all $\boldsymbol{Y}_0$.

Proof: See Appendix 7.

3.1.2 Regeneration and Explicit Regeneration Times

Recall that a Markov chain with an invariant probability measure admits coupling if and only if it is Harris ergodic. In the previous subsection we used a coupling argument for $\boldsymbol{Y}$, therefore it is now easy to show that,

Theorem 3. Let demands $\{\boldsymbol{D}_n, n \geq 0\}$ be i.i.d. with $\mathbf{E}[||\boldsymbol{D}_0||] < \min_{k,m}\{C^{km}\}$. Then $\{\boldsymbol{Y}_n, n \geq 0\}$ is a Harris ergodic Markov chain.

Proof: Since $\boldsymbol{Y}_{n+1} = \Phi(\boldsymbol{Y}_n, \boldsymbol{D}_n)$, $n \geq 0$, $\boldsymbol{Y}$ is a Markov chain when $\boldsymbol{D}$ is i.i.d. We established in Theorem 1 and Corollary 1 that $\boldsymbol{Y}$ has an invariant (i.e., stationary) distribution and in Theorem 2 that $\boldsymbol{Y}$ admits coupling. Thus, $\boldsymbol{Y}$ is Harris ergodic.

QED

As a result of Theorem 3, $\boldsymbol{Y}$ inherits the regenerative structure of Harris ergodic Markov chains, the attendant ratio formula, and convergence results. The same holds for the inventory levels:

Corollary 2. The inventory process $\{(\boldsymbol{I}_n^{11}, \ldots, \boldsymbol{I}_n^{KM}), n \geq 0\}$, under the conditions of Theorem 3, is a Harris ergodic Markov chain.

Proof: There is a one-to-one correspondence between shortfalls and inventories for all n as defined by

$$\begin{aligned} I_n^{11p} &= z^{11p} - Y_n^{11p} \\ I_n^{kmp} &= (z^{kmp} - z^{(km)^- p}) + (Y_n^{(km)^- p} - Y_n^{kmp}). \end{aligned} \tag{14}$$

Consequently, $\boldsymbol{I} = \{\boldsymbol{I}_n, n \geq 0\}$ is Markov if $\boldsymbol{Y}$ is, and $\boldsymbol{I}$ is Harris ergodic if $\boldsymbol{Y}$ is.

QED

It is now possible to give the characterization of the regeneration times.

Theorem 4. Let demands be i.i.d. with $\mathbf{E}[||\boldsymbol{D}_0||] < \min_{k,m}\{C^{km}\}$. Define $\boldsymbol{z}^{(11)^-} \equiv \mathbf{0}$ and suppose that

$$P(d_0^p \leq z^{kmp} - z^{(km)^- p}) > 0, \;\; k = 1, \dots, K; \; m = 1, \dots, M; \; p = 1, \dots, P. \tag{15}$$

Then $\boldsymbol{Y}$ returns to the origin infinitely often, with probability one.

Proof: See Appendix 7.

Corollary 3. The inventory process $\{(\boldsymbol{I}_n^{11}, \dots, \boldsymbol{I}_n^{KM}), n \geq 0\}$, under the conditions of Theorem 4, returns to $(\boldsymbol{z}^{11}, \boldsymbol{z}^{(11)^+} - \boldsymbol{z}^{11}, \dots, \boldsymbol{z}^{KM} - \boldsymbol{z}^{(KM)^-})$ infinitely often, with probability one.

Proof: Consequence of the relationship between shortfalls and inventories.

QED

The conclusion of Theorem 4 is not in general true without (15) or further distributional assumptions on demands. This is particularly clear when $z^{(km)^+ p} = z^{kmp}$ for some value of k, m, and p; that is, stage $(km)^+$ keeps no safety stock for product p. In this case, the total shortfall $||\boldsymbol{Y}^{km}||$ can never reach zero unless $d_0^p = 0$ with positive probability.

3.2 *Stability and Regeneration for Totally Shared Systems*

Operating on a TS mode with the LSR, the production decision is given by

$$P_n^{kmp} = f_n^{kmp} g_n^m, \tag{16}$$

capacity is shared among all products and levels for each machine, and g_n^m is given by (17).

$$g_n^m = \min\left\{ \frac{C^m}{\sum_{k=1}^K \sum_{p=1}^P f_n^{kmp}}, 1 \right\}. \tag{17}$$

For this model the shortfall process is governed by the following

$$\begin{aligned} \boldsymbol{Y}_{n+1}^{km} = \max\{&\boldsymbol{Y}_n^{km} + \boldsymbol{D}_n - \frac{C^m \min\{\boldsymbol{Y}_n^{km} + \boldsymbol{D}_n, \boldsymbol{I}_n^{(km)+}\}}{\sum_{k=1}^K ||\min\{\boldsymbol{Y}_n^{km} + \boldsymbol{D}_n, \boldsymbol{I}_n^{(km)^+}\}||}, \\ &\boldsymbol{0}, \boldsymbol{Y}_n^{km} + \boldsymbol{D}_n - \boldsymbol{I}_n^{(km)^+}\} \end{aligned} \tag{18}$$

Due to the structure of the decision process, it is the case that

$$\begin{aligned} ||\boldsymbol{Y}_{n+1}^{km}|| = \max\{0, ||\boldsymbol{Y}_n^{km}|| + ||\boldsymbol{D}_n|| - \frac{C^m ||\min\{\boldsymbol{Y}_n^{km} + \boldsymbol{D}_n, \boldsymbol{I}_n^{(km)^+}\}||}{\sum_{k=1}^K ||\min\{\boldsymbol{Y}_n^{km} + \boldsymbol{D}_n, \boldsymbol{I}_n^{(km)^+}\}||}, \\ \sum_{p=1}^P \left(Y_n^{(km)^+ p} + d_n^p - (z^{(km)^+ p} - z^{kmp}) \right)^+ \}, \end{aligned} \tag{19}$$

where it is assumed that $\boldsymbol{I}_n^{(KM)^+} = \infty$ for all n.

As in the previous model we were interested on the total shortfall per level and stage, we will now be interested in the *Total Shortfall per Stage*. Thus, the following expression is of importance in what follows

$$\begin{aligned} \sum_{k=1}^K ||\boldsymbol{Y}_{n+1}^{km}|| = \max\{0, &\sum_{k=1}^K ||\boldsymbol{Y}_n^{km}|| + K||\boldsymbol{D}_n|| - C^m, \\ &\sum_{k=1}^K \sum_{p=1}^P \left(Y_n^{(km)^+ p} + d_n^p - (z^{(km)^+ p} - z^{kmp}) \right)^+ \} \end{aligned} \tag{20}$$

3.2.1 The Stationary Regime

Lemma 3. The total echelon shortfall per stage satisfies

$$\sum_{k=1}^K ||\boldsymbol{Y}_{n+1}^{km}|| = \psi(\boldsymbol{Y}_n, \boldsymbol{D}_n) = \sum_{k=1}^K \phi(\boldsymbol{Y}_n, \boldsymbol{D}_n), \tag{21}$$

where $\phi : R_+^{KMP} \times R^P \to R_+^{KM}$ is defined by (19) and $\psi : R_+^{KMP} \times R^P \to R_+^M$ is defined by (20). In particular, ψ is continuous and increasing.

It is easily possible to establish a result similar to that of Lemma 2 for this second model and to prove stability we will make use of Theorem 1.

Theorem 5. Under the assumptions of Theorem 1, the system operated under the TS mode is stable, in the sense that the shortfalls are almost surely finite, if

$$K\mathbf{E}[||\boldsymbol{D}_0||] < \min\{C^m : m = 1, \ldots, M\}. \tag{22}$$

Proof: Assume that the capacity of each machine is divided into slots of equal size, that is $C^{km} = \frac{C^m}{K}$. Assume the system operates as if capacity was not shared. Then, according to Theorem 1, the system would be stable iff

$$\begin{aligned}\mathbf{E}[||\boldsymbol{D}_0||] &< \min\{C^{km} : k = 1, \ldots, K, m = 1, \ldots, M\} \\ &= \frac{1}{K}\min\{C^m : m = 1, \ldots, M\}\end{aligned}$$

Now we have to evaluate how does the system behave under the TS case when compared with its performance under the PS case. To show that stability of the PS case implies stability of the TS case we will investigate a sample path.

Assume we have two identical systems subject to the same sample path. One is operated under the PS mode with $C^{km} = \frac{C^m}{K}$ and the other is operated under the TS mode. In particular, one is interested in the process defined by the total shortfall for each stage. Assume that both systems start from the origin, that is

$$\sum_{k=1}^{K} ||\boldsymbol{Y}_0^{km}||^1 = \sum_{k=1}^{K} ||\boldsymbol{Y}_0^{km}||^2 = 0 \text{ for all } k, m. \tag{23}$$

Comparing equation (19) with (10) and (12) it is obvious that while there is no bound in capacity for any of the two systems they remain coupled. Let n^* denote the first period for which at least one of the two systems has a bound in capacity for some level and/or stage. Therefore, we have for all k and m

$$||\boldsymbol{Y}_n^{km}||^1 = ||\boldsymbol{Y}_n^{km}||^2 \text{ for all } n = 0, \ldots, n^*, \tag{24}$$

which implies that

$$\sum_{k=1}^{K} ||\boldsymbol{Y}_n^{km}||^1 = \sum_{k=1}^{K} ||\boldsymbol{Y}_n^{km}||^2 \text{ for all } n = 0, \ldots, n^*. \tag{25}$$

The first time one of these two systems has at least one production decision bounded by capacity there is a possibility for decoupling. Let us take system 1 as the reference. Whenever there is at least a level and stage for which system 1 is bound by capacity, one of two things can happen to system 2:

i) Bound in capacity for system 1 and no bound for system 2.
For this case there exists at least a k^* and an m^* such that

$$||\min\{\boldsymbol{Y}_{n^*}^{k^*m^*} + \boldsymbol{D}_{n^*}, \boldsymbol{I}_{n^*}^{(k^*m^*)^+}\}|| > \frac{C^{m^*}}{K}, \tag{26}$$

but

$$\sum_{k=1}^{K} ||\min\{Y_{n^*}^{km^*} + D_{n^*}, I_{n^*}^{(km^*)^+}\}|| < C^{m^*}. \tag{27}$$

ii) Bound in capacity for both systems
In this case we have at least a k^* and an m^* such that

$$||\min\{\boldsymbol{Y}_{n^*}^{k^*m^*} + \boldsymbol{D}_{n^*}, \boldsymbol{I}_{n^*}^{(k^*m^*)^+}\}|| > \frac{C^{m^*}}{K}, \tag{28}$$

and

$$\sum_{k=1}^{K} ||\min\{\boldsymbol{Y}_{n^*}^{km^*} + \boldsymbol{D}_{n^*}, \boldsymbol{I}_{n^*}^{(km^*)^+}\}|| > C^{m^*}. \tag{29}$$

We are interested on knowing how does $\sum_{k=1}^{K} ||\boldsymbol{Y}_n^{km}||$ (the total shortfall for stage m) behave for both cases. In case i) since system 2 has no bound in capacity it must be the case that not all the levels of stage m^* have a bound in capacity for system 1. Therefore,

$$\begin{aligned}
\sum_{k=1}^{K} ||\boldsymbol{Y}_{n^*+1}^{km^*}||^1 &= \sum_{k\neq k^*} \sum_{p=1}^{P} \left(Y_{n^*}^{(km^*)^+ p} + d_{n^*}^p - (z^{(km^*)^+ p} - z^{km^* p}) \right)^+ + \\
&\quad + \sum_{k=k^*} (||\boldsymbol{Y}_{n^*}^{km^*}||^1 + ||\boldsymbol{D}_{n^*}|| - \frac{C^{m^*}}{K}) \\
&> \sum_{k=1}^{K} \sum_{p=1}^{P} \left(Y_{n^*}^{(km^*)^+ p} + d_{n^*}^p - (z^{(km^*)^+ p} - z^{km^* p}) \right)^+ \\
&= \sum_{k=1}^{K} ||\boldsymbol{Y}_{n^*+1}^{km^*}||^2
\end{aligned} \tag{30}$$

because there is a bound in capacity for all levels k^* in system 1 and there is no such bound in system 2 and using equations (12) and (19)[2].

For case ii), when both systems are capacity bounded for some stage m^*, there are two possibilities: there is a bound in capacity for all levels of stage m^* in system 1; not all levels of stage m^* are capacity bounded for system 1.

For the first situation it will be the case that

$$\sum_{k=1}^{K} ||\boldsymbol{Y}_{n^*+1}^{km^*}||^1 = \sum_{k=1}^{K} (||\boldsymbol{Y}_{n^*}^{km^*}||^1 + ||\boldsymbol{D}_{n^*}|| - \frac{C^{m^*}}{K})$$

[2] Note that equation (10) can be made equal to (12) by defining $I_n^{(KM)^+} = \infty$ and making the adequate change for $z^{(KM)^+}$.

$$= \sum_{k=1}^{K} ||\boldsymbol{Y}_{n^*+1}^{km^*}||^2, \tag{31}$$

because $\sum_{k=1}^{K} ||\boldsymbol{Y}_{n^*}^{km*}||^1 = \sum_{k=1}^{K} ||\boldsymbol{Y}_{n^*}^{km*}||^2$.

In the second situation we will have

$$\begin{aligned}
\sum_{k=1}^{K} ||\boldsymbol{Y}_{n^*+1}^{km^*}||^1 &= \sum_{k \neq k^*} \sum_{p=1}^{P} \left(Y_{n^*}^{(km^*)^+ p} + d_{n^*}^{p} - (z^{(km^*)^+ p} - z^{km^* p}) \right)^+ + \\
&+ \sum_{k=k^*} (||\boldsymbol{Y}_{n^*}^{km^*}||^1 + ||\boldsymbol{D}_{n^*}|| - \frac{C^{m^*}}{K}) \\
&> \sum_{k=1}^{K} ||\boldsymbol{Y}_{n^*}^{km^*}||^1 + K||\boldsymbol{D}_{n^*}|| - C^{m^*} \\
&= \sum_{k=1}^{K} ||\boldsymbol{Y}_{n^*+1}^{km^*}||^2
\end{aligned} \tag{32}$$

because the change in total shortfall for stage m^* in system 1 is smaller than C^{m^*}.

Thus, we have that for period n^*+1 the total shortfall for each stage of system 2 is bounded above by the total shortfall for each stage of system 1, with probability one.

Now it remains to see what happens after n^*+1 (the first decoupling period). Assume, there is a third system that starts operating in a TS mode as system 2 with the state variables of system 1, that is, coupled to system 1. System 1 and system 3 will remain coupled until a capacity bound occurs at some other period. By the above discussion we know that a bound in capacity is favorable to system 3, when compared with system 1. Due to Lemma 3 the total shortfall per stage of system 2 will remain dominated by that of system 3. So we have that until the first decoupling between system 1 and system 3, system 1 will dominate system 2, due to transitivity. If we force system 3 to receive the state of system 1 whenever there is a decoupling between the two the process repeats itself whenever there is a new bound in capacity and it follows then that the total shortfall per stage for system 2 will remain dominated by that of system 1, with probability one.

QED

In order to establish the uniqueness of the distribution it is also possible to show that the total shortfall per stage process admits coupling.

Theorem 6. Under the stability condition $K\mathbf{E}[||\boldsymbol{D}_0||] < \min_m\{C^m\}$, the total shortfall per stage admits coupling and so does the shortfall process as a consequence. Therefore, its stationary distribution is unique and $\boldsymbol{Y}_n \Rightarrow \tilde{\boldsymbol{Y}}_0$ for all $\boldsymbol{Y}_0$.

Proof: According to the proof of Theorem 5 the total shortfall process per stage of the PS case dominates that of the TS case. Therefore, if the first admits coupling, so does the second because the shortfalls are always non negative. By Theorem 2, it is the case that the first admits coupling.

Thus, the result follows.

QED

3.2.2 Regeneration and Explicit Regeneration Times

Since in the previous subsection a coupling argument was used for Y, it is now easy to show the following.

Theorem 7. Let demands $\{\boldsymbol{D}_n, n \geq 0\}$ be i.i.d. with $K\mathbf{E}[||\boldsymbol{D}_0||] < \min_m\{C^m\}$. Then $\{\boldsymbol{Y}_n, n \geq 0\}$ is a Harris ergodic Markov chain.

Proof: Since $\sum_{k=1}^{K} ||\boldsymbol{Y}_{n+1}^{km}|| = \psi(\boldsymbol{Y}_n, \boldsymbol{D}_n)$, $n \geq 0$, $\boldsymbol{Y}$ is a Markov chain when $\boldsymbol{D}$ is i.i.d. Theorem 5 established that $\boldsymbol{Y}$ has an invariant (i.e., stationary) distribution and Theorem 6 established that $\boldsymbol{Y}$ admits coupling. Thus, $\boldsymbol{Y}$ is Harris ergodic.

QED

Corollary 4. The inventory process $\{(\boldsymbol{I}_n^{11}, \ldots, \boldsymbol{I}_n^{KM}), n \geq 0\}$, under the conditions of Theorem 3, is a Harris ergodic Markov chain.

Proof: There is a one-to-one correspondence between shortfalls and inventories for all n. Consequently, $\boldsymbol{I} = \{\boldsymbol{I}_n, n \geq 0\}$ is Markov if $\boldsymbol{Y}$ is, and $\boldsymbol{I}$ is Harris ergodic if $\boldsymbol{Y}$ is.

QED

The regeneration times can now be characterized.

Theorem 8. Let demands be i.i.d. with $K\mathbf{E}[||\boldsymbol{D}_0||] < \min_m\{C^m\}$. Define $\boldsymbol{z}^{(11)^-} \equiv \mathbf{0}$ and suppose that

$$P(d_0^p \leq z^{kmp} - z^{(km)^- p}) > 0, \quad k = 1, \ldots, K; \ m = 1, \ldots, M; \ p = 1, \ldots, P. \tag{33}$$

Then $\boldsymbol{Y}$ returns to the origin infinitely often, with probability one.

Proof: The proof follows from the fact that the same system operated under a PS mode with $C^{km} = C^m/K$ will have a shortfall process that dominates that of a system operated on a TS mode. Since for the PS mode Theorem 4 is applicable it is the case that if $\boldsymbol{Y}$ returns to origin infinitely often under the PS mode so it does for the TS mode due to the dominance earlier discussed.

QED

Corollary 5. The inventory process $\{(\boldsymbol{I}_n^{11}, \ldots, \boldsymbol{I}_n^{KM}), n \geq 0\}$, under the conditions of Theorem 4, returns to $(\boldsymbol{z}^{11}, \boldsymbol{z}^{(11)^+} - \boldsymbol{z}^{11}, \ldots, \boldsymbol{z}^{KM} - \boldsymbol{z}^{(KM)^-})$ infinitely often, with probability one.

Proof: Consequence of the relationship between shortfall variables and inventories.

QED

4 Non Uniform Loads and Perfect Yield

We review the models presented in Section 1 to accommodate this extra feature. The recursions for inventory, echelon inventory, and shortfall do not change. What changes are the specifics of the production decisions. Recall that the production expression for the LSR operated in the PS mode is

$$P_n^{kmp} = f_n^{kmp} g_n^{km}. \tag{34}$$

Since the net production request, f_n^{kmp}, only depend on shortfalls and feeding inventories their expressions do not change when including the non uniform loads. What changes is the expression for g_n^{km}, because it accounts for the impact of the net request over the available capacity. Let us assume that every product p on level k and stage m needs τ^{kmp} units of capacity per unit of material produced. In the analysis so far it was assumed that $\tau^{kmp} = 1$ for all k,m,p. Given the inclusion of the τ^{kmp} constants, not necessarily all equal to 1, the expression for g_n^{km} becomes:

$$g_n^{km} = \min\left\{ \frac{C^{km}}{\sum_p \tau^{kmp} f_n^{kmp}}, 1 \right\} \tag{35}$$

Whereas f_n^{kmp} expresses the net production request in terms of parts, the term $\tau^{kmp} f_n^{kmp}$ expresses that request in terms of machine capacity.

As before, let us first address the discussion of stability for the PS mode.

4.1 Stability and Regeneration for Partially Shared Systems

For this setting there is no substantial change relative to the partially shared systems with perfect yield and uniform loads by replacing $||\boldsymbol{Y}_n^{km}||$ with $||\boldsymbol{Y}_n^{km}||_\tau$, defined as

$$||\boldsymbol{Y}_n^{km}||_\tau = \sum_{p=1}^{P} \tau^{kmp} Y_n^{kmp}. \tag{36}$$

With this change, equation (10) becomes

$$\begin{aligned} ||\boldsymbol{Y}^{KM}_{n+1}||_\tau &= \max\{0, ||\boldsymbol{Y}^{KM}_n + \boldsymbol{D}_n||_\tau (1 - \frac{C^{KM}}{||\boldsymbol{Y}^{KM}_n + \boldsymbol{D}_n||_\tau})\} \\ &= \max\{0, ||\boldsymbol{Y}^{KM}_n||_\tau + ||\boldsymbol{D}_n||_\tau - C^{KM}\} \end{aligned} \tag{37}$$

and equation (12) becomes

$$\begin{aligned} ||\boldsymbol{Y}^{km}_{n+1}||_\tau = \max \Big\{ & 0, ||\boldsymbol{Y}^{km}_n||_\tau + ||\boldsymbol{D}_n||_\tau - C^{km}, \\ & \sum_{p=1}^{P} \tau^{kmp} \left(Y_n^{(km)^+ p} + d_n^p - (z^{(km)^+ p} - z^{kmp}) \right)^+ \Big\}. \end{aligned} \tag{38}$$

These dynamic equations, for the *weighted shortfall sums*, fall exactly into the framework described in Section 3.1. Therefore, the adequate stability condition becomes the following.

Theorem 9. Suppose the demands $\{\boldsymbol{D}_n, -\infty < n < \infty\}$ are ergodic as well as stationary. If

$$\mathbf{E}[||\boldsymbol{D}_0||_\tau] = \sum_{p=1}^{P} \tau^{kmp} \mathbf{E}[d_0^p] < C^{km} \quad \text{for all k, m.} \tag{39}$$

then the shortfall process is stable when the system is operated in the PS mode.

Proof: After performing the changes above indicated, the proof is the same as that of Theorem 1.

QED

All the results presented for the PS mode in Section 3.1 are valid for this setting without change.

4.2 Stability and Regeneration for Totally Shared Systems

A simple observation of equations (37) and (38) helps to understand why we cannot resort to the technique used in Section 3.2, when proving stability for totally shared systems with perfect yield and uniform loads. Note that the stochastic dominance may be destroyed when production is bound by inventory. When loads are uniform, all values of $\tau^{kmp} = 1$ and stochastic dominance follows trivially. This dominance would be maintained if the value $Y_n^{(km)^+ p}$ would be multiplied by $\tau^{(km)^+ p}$ in the expressions above, but it is multiplied by τ^{kmp}. In general $\tau^{kmp} \neq \tau^{(km)^+ p}$.

For totally shared capacity systems there is a need to introduce some changes on the structure of the control policies. The stability will be established by presenting a particular choice of parameters for the new control policy that yields a stable system. Given the proposed choice of parameters is feasible and induces stability, it necessarily constitutes an upper bound on the cost. The optimal parameters will have to incur lower costs. Therefore, by providing an upper bound which is stable the stability of the system will be asserted.

The main structural change on the control policies proposed is the addition of an input bound. That is, there is a need to impose a maximum amount of new material entering the production system for each product per period. Although some bound exists already, given that machines have finite capacity, this is not enough to establish stability. It is necessary to define tighter bounds. In [6] the PR performs quite poorly when the entering level has priority over the others. Also, it was shown that the degradation of the LSR when switching from the PS mode to the TS mode is due to the fact that the potential input of new material jumps from C^{KM} to a total of C^{M} per period, distorting the proportions between the several levels in favor of the input of new material. This preference is given at the expense of a slower travel speed along the production line. Moreover, it was shown that in the PS mode a system with $K=2$, $M=P=1$, and $C^{21}>C^{11}$ improves its performance if we chop the excess capacity of level 2, making $C^{21}=C^{11}$. Then it was argued that having a higher capacity on level 2 only increases the speed at which inventory moves to the buffer feeding level 1, but does not make it move faster towards the output buffer, since level 1 is the bottleneck.

Although stability is not at risk for the cases discussed in [6], the fact that we could benefit from the existence of an input bound in such cases constitutes strong evidence favoring the definition of this richer class of control policies. Besides having the base stock variables as the control parameters, we can have the input bound as an additional control variable, thus defining a wider class of multi-echelon base stock policies. The existence of such bounds is crucial to establish stability.

Therefore, we resort to class Π_1 policies and define a set of parameters that stabilizes any of our re-entrant systems for any of the proposed production rules. Let

$$\Delta^{p}=\min_{k,m}\{\frac{\mathbf{E}[d_0^p]}{\sum_{i=1}^{K}\sum_{j=1}^{P}\tau^{imj}\mathbf{E}[d_0^j]}C^m\} \quad \text{for all } p=1,\dots,P, \tag{40}$$

and define $I^{(KM)^+p}=\Delta^p$ as the bound for the input of product p into the system. That is, $I^{(KM)^+p}$ is the feeding inventory of stage M and level K. Set $\Delta^{kmp}=\Delta^p$ for all k and m, except for Δ^{11p} that may assume any positive value.

Assume that the system is operated using any production rule in the TS mode. With this set of delta variables all inventory variables, except I_n^{11p}, will always be Δ^p for each product. At any level and stage, the amount

$$\sum_{k=1}^{K}\sum_{p=1}^{P}\tau^{kmp}\Delta^p\le C^m, \tag{41}$$

by the definition of Δ^p. Therefore, there is never a bound in capacity and the system behaves as if there is no capacity sharing, thus being operated as if there exist P different and decoupled production systems with no re-entrance. The only bound in capacity occurs for the equality between net production request and capacity which can be seen as a no capacity bound situation, since the match is perfect.

We know that for no sharing of capacity a system is stable as long as $\tau^{kmp}\mathbf{E}[d_0^p] < C^{kmp}$. This conclusion is easily derived from the stability result for partially shared systems with single product, discussed before (Theorem 9). Slicing the capacity of machine m into $k \times p$ slots and calling each one C^{kmp}, for $k = 1,\dots,K$ and $p = 1,\dots,P$ and adding over all products and levels we get

$$\sum_{k=1}^{K}\sum_{p=1}^{P}\tau^{kmp}\mathbf{E}[d_0^p] < \sum_{k=1}^{K}\sum_{p=1}^{P}C^{kmp} = C^m, \tag{42}$$

which is the stability condition for totally shared systems with non uniform loads and perfect yield. This condition holds iff $\mathbf{E}[d_0^p] < \Delta^p$ for all $p = 1,\dots,P$.

Having provided a set of parameters which stabilizes the production system for any production rule in the TS mode it should now be evident that the optimal set of parameters will have to incur lower costs than the costs incurred by the parameters just defined. The optimal set of parameters cannot, therefore, induce an unstable system as long as $\Delta^p > \mathbf{E}[d_0^p]$. The following result has been proven.

Theorem 10. Suppose the demands $\{\boldsymbol{D}_n, -\infty < n < \infty\}$ are ergodic as well as stationary. If (42) holds, then the shortfall process is stable when the system is operated in the TS mode, using class Π_1.

The regeneration and explicit regeneration times discussed earlier carry through trivially for this setting.

4.2.1 Remarks on the Class Π_1

For the system to be stable, the minimum amount of each product that can get through the system at any period has to be above the average demand. This is the same as saying that the bottleneck machine, the machine for which (41) holds in the equality, has capacity above the load imposed by the demand process.

Note that one can use any of the production rules and, in the particular case of the priority rule, one can use any arbitrary priority list without risking stability. This constitutes a strength of the class of policies introduced (recall the literature review on stability).

Moreover, the argument here used for stability allows us to drop one of the main constraints of the present model: the re-entrant structure adopted. This technique extends easily to more complex re-entrant systems where not all the products are processed by the same number of levels and not all the products visit all the machines in the same order. Such was not the case of the stability proof for systems with uniform loads, since the stochastic dominance argument relies on the fact that the shortfalls added belong to the output buffers of the same machines.

The optimal policy does not necessarily have the above bound for the entering inventory. It may be the case that, during the optimization, the solution converges to values of $I^{(KM)^+p}$ which are equal or above C^M for all $p = 1,\dots,P$. If such is the case we may drop the explicit bound on input inventory, since being above C^M has no physical significance. The cases where the optimization procedure converges to values of $I^{(KM)^+p}$ below C^M, can be clearly identified as systems that may need such bound for the input inventory in order to remain stable. Naturally, it is not necessarily the case that all the systems for which the optimal $I^{(KM)^+p}$ is under C^M are only stabilized by policies from class Π_1, since cost considerations are taken into account when determining such values. Note also that while policies in Π_0 are nonidling in terms of the shortfalls, such is no longer the case for policies in Π_1.

5 Uniform Loads and Random Yield

To accommodate random yield we simply change the dynamic equations for inventories and for shortfalls. The multiplicative random yield, α_n^{kmp}, is assumed to be independent for each level, stage, and product. Also, it is assumed that the random yield is continuous and i.i.d. for each period taking values in the set $[0,1]$. Demands are assumed continuous, independent across products, and i.i.d. for each product along time. Both sets of random variables, demand and yield, are assumed independent.

The shortfall dynamic equation in the presence of random yield assumes the following form:

$$Y_{n+1}^{kmp} = Y_n^{kmp} + d_n^p - \alpha_n^{kmp} P_n^{kmp} + \sum_{qr=(km)^-}^{q,r=1,1} (1-\alpha_n^{qrp}) P_n^{qrp}, \tag{43}$$

where the additional summation accounts for the parts lost in the downstream machines due to the presence of random yield.

For the random yield case it is easy to show stability for single product, NS mode, with uniform or non uniform loads. To prove stability for the multiple product cases and other sharing schemes we follow the approach of Section 4.

5.1 *Stability and Regeneration for Partially Shared Systems*

The presence of random yield in the context of uniform loads does not change the basics of the formal result. The main difference is the explicit stability condition. Aside from that, we can repeat the same steps as in Section 3.1. Therefore, the stability condition proof will be presented and the natural extension of previous results to this situation will be listed.

Assume a system operating in the PS mode with the LSR and replace P_n^{kmp} in the dynamic equation for the shortfall variables.

$$
\begin{aligned}
Y_{n+1}^{kmp} &= Y_n^{kmp} + d_n^p + \sum_{qr=(km)^-}^{q,r=1,1} (1-\alpha_n^{qrp}) P_n^{qrp} - \alpha_n^{kmp} \min\{ f_n^{kmp}, f_n^{kmp} \frac{C^{km}}{\sum_{p=1}^{P} f_n^{kmp}} \} \\
&= \max\{ Y_n^{kmp} + d_n^p + \sum_{qr=(km)^-}^{q,r=1,1} (1-\alpha_n^{qrp}) P_n^{qrp} - \alpha_n^{kmp} f_n^{kmp}, \\
&\qquad Y_n^{kmp} + d_n^p + \sum_{qr=(km)^-}^{q,r=1,1} (1-\alpha_n^{qrp}) P_n^{qrp} - \alpha_n^{kmp} f_n^{kmp} \frac{C^{km}}{\sum_{p=1}^{P} f_n^{kmp}} \}
\end{aligned}
\tag{44}
$$

where $f_n^{kmp} = \min\{Y_n^{kmp} + d_n^p, I_n^{(km)^+ p}\}$.

The above dynamic equation for the shortfall variables is not as easy to deal with as it was for previous settings. Because of this, one has to proceed differently. First, the stability condition for single product systems with no re-entrance is established. Later, by the approach of Section 4, stability for the PS mode for multiple products will be defined. We show that the stability condition for the PS mode is

$$
\sum_{p=1}^{P} \frac{\mathbf{E}[d_0^p]}{\prod_{q,r=1,1}^{q,r=k,m} \mathbf{E}[\alpha_0^{qrp}]} < C^{km} \quad \text{for} \begin{cases} m = 1, \ldots, M \\ k = 1, \ldots, K \end{cases}
\tag{45}
$$

The indexes in $\prod_{q,r=1,1}^{q,r=k,m} h(q,r)$ signify that the factors are taken up the production line from $h(1,1)$ to $h(k,m)$. It does not mean that the iteration is taken from 1 to k and from 1 to m independently of each other.

To simplify the notation, consider a system with single product and no re-entrance in the presence of random yield and composed of M machines. Except for random yield, this is addressed by [13, 14]; we add random yield here. For this simplified version, we have

Theorem 11. Suppose the demand $\{d_n, -\infty < n < \infty\}$ is ergodic as well as stationary. Additionally, suppose the random yield $\{\alpha_n^m, -\infty < n < \infty\}$ is ergodic and stationary. The shortfall process is stable iff

$$
\frac{\mathbf{E}[d_0]}{\prod_{i=1}^{i=m} \mathbf{E}[\alpha_0^i]} < C^m, \quad \text{for all } m = 1, \ldots, M,
\tag{46}
$$

holds for the single product system.

Proof: The dynamic equation for shortfalls will be

$$
\begin{aligned}
Y_{n+1}^m = \max\{ & Y_n^m + d_n + \sum_{i=m-1}^{1} (1-\alpha_n^i) P_n^i - \alpha_n^m f_n^m, \\
& Y_n^m + d_n + \sum_{i=m-1}^{1} (1-\alpha_n^i) P_n^i - \alpha_n^m C^m \}
\end{aligned}
\tag{47}
$$

which, by direct comparison with the equation for perfect yield leads to the following necessary and sufficient stability condition

$$\mathbf{E}[d_0 + \sum_{i=m-1}^{1} (1-\alpha_0^i)P^i] < \mathbf{E}[\alpha_0^m C^m]. \tag{48}$$

In the perfect yield situation, it holds that $\alpha_n^{kmp} = 1$ and it is the case that the system is stable iff $\mathbf{E}[d_n - C^m] < 0$. A similar reasoning is applied here to propose the above condition: this condition ensures the existence of a negative drift when production is bound by capacity. We need only to establish a connection between (46) and (48). To do so, we first establish a relationship between production amounts in consecutive machines.

The production of machine i is conditioned by what is effectively produced by machine $(i+1)$. What is effectively produced by machine $(i+1)$ during period n is $\alpha_n^{i+1}P_n^{i+1}$. If production starts at a point where $I_0^{i+1} = \Delta^{i+1} = z^{i+1} - z^i$ it turns out that

$$\sum_{n=1}^{N} P_n^i \leq \Delta^{i+1} + \sum_{n=1}^{N} \alpha_n^{i+1} P_n^{i+1}, \tag{49}$$

since machine i cannot engage more material in production than the available inventory.

Dividing the above by N and taking the limit as $N \to \infty$ we get

$$\mathbf{E}[P^i] \leq \mathbf{E}[\alpha_0^{i+1}]\mathbf{E}[P^{i+1}]. \tag{50}$$

Given that α_n^{i+1} is independent of P_n^{i+1}, the yield process is i.i.d., and the machines are capacitated, the limit exists and equals the expected value.

Assume now that the inequality above holds strictly. If that is the case, the inventory sitting in front of machine i, I^{i+1}, grows to infinity because

$$I_n^{i+1} = \Delta^{i+1} + \sum_{j=1}^{n} (\alpha_j^{i+1} P_j^{i+1} - P_j^i), \tag{51}$$

and taking the limit as $n \to \infty$ we get

$$\begin{aligned} I_\infty^{i+1} &= \Delta^{i+1} + \lim_{n\to\infty} \sum_{j=1}^{n} (\alpha_j^{i+1} P_j^{i+1} - P_j^i) \\ &= \Delta^{i+1} + \lim_{n\to\infty} \sum_{j=1}^{n} \alpha_j^{i+1} P_j^{i+1} - \lim_{n\to\infty} \sum_{j=1}^{n} P_j^i \\ &= \Delta^{i+1} + \lim_{n\to\infty} n\mathbf{E}[\alpha^{i+1}]\mathbf{E}[P^{i+1}] - \lim_{n\to\infty} n\mathbf{E}[P^i] \\ &= \infty, \end{aligned} \tag{52}$$

by the law of large numbers and because of the assumption on the strict inequality.

If the value of the feeding inventory for any machine grows to infinity the system is unstable. Also, if the value of the feeding inventory grows to infinity, it must be the case that production of that machine is being bound by capacity in the long run. Thus, it is established that on a stable system it must be the case that (50) holds at equality for all machines. It is also easy to show that

$$\mathbf{E}[P^i] \leq C^i \quad \text{for } i = 1, \ldots, M. \tag{53}$$

The expected production of any machine is either bounded by the expected production of the preceding machine as presented in (50) or is bounded by the available capacity as presented in (53). That is, only one of these inequalities will hold at equality. If at least for one machine the bound occurs due to capacity, then the system is unstable, implying $\mathbf{E}[\alpha_0^1]\mathbf{E}[P^1] < \mathbf{E}[d_0]$ and the value of I^1 grows to $-\infty$. For a system to track demand, $\mathbf{E}[\alpha_0^1]\mathbf{E}[P^1]$ has to be equal to $\mathbf{E}[d_0]$.

Now, observe that if all $m-1$ stages are stable, each $\mathbf{E}[P^i]$, for $i = 1, \ldots, m-1$, can be written as a function of $\mathbf{E}[P^1]$ as follows[3]:

$$\mathbf{E}[P^i] = \frac{\mathbf{E}[P^1]}{\prod_{j=2}^{i} \mathbf{E}[\alpha_0^j]}, \tag{54}$$

and $\mathbf{E}[P^1] = \frac{\mathbf{E}[d_0]}{\mathbf{E}[\alpha_0^1]}$.

Proceeding by backward induction, consider first the case of $m = 1$. Expression (48) will reduce to

$$\mathbf{E}[d_0] < \mathbf{E}[\alpha_0^1]C^1, \tag{55}$$

which is exactly the same as (46) for $m = 1$. To prove the stability condition for stage m, let us assume that all $m-1$ downstream stages are stable. That is, assume that instability cannot be caused by the last $m-1$ machines. Therefore, (48) becomes

$$\begin{aligned}
\mathbf{E}[\alpha_0^m]C^m &> \mathbf{E}[d] + \mathbf{E}[1-\alpha^{m-1}]\frac{\mathbf{E}[P^1]}{\prod_{j=2}^{m-1}\mathbf{E}[\alpha^j]} + \ldots \mathbf{E}[1-\alpha^1]\mathbf{E}[P^1] \\
&= \mathbf{E}[d] + \mathbf{E}[P^1]\left(\frac{1}{\prod_{j=2}^{m-1}\mathbf{E}[\alpha^j]} - \mathbf{E}[\alpha^1]\right) \\
&= \mathbf{E}[d] + \mathbf{E}[d]\left(\frac{1}{\prod_{j=1}^{m-1}\mathbf{E}[\alpha^j]} - 1\right) \\
&= \mathbf{E}[d]\frac{1}{\prod_{j=1}^{m-1}\mathbf{E}[\alpha^j]}
\end{aligned} \tag{56}$$

showing that if (48) holds, so does (46). It remains to see what happens when (48) does not hold.

[3] Using (50) with the equality sign.

Let us assume that (48) does not hold for at least one machine. Given that this is a necessary and sufficient condition for stability, it follows that the system is unstable. Therefore, it must be the case that $\mathbf{E}[P^1] < \mathbf{E}[d_0]/\mathbf{E}[\alpha_0^1]$.

Given that there is at least one machine violating (48), let m^* be the bottleneck machine of the line. That is, the machine that is furthest away from the stability region. For this machine it is the case that $\mathbf{E}[P^{m^*}] = C^{m^*}$ and for all the machines downstream it is the case that (50) holds at equality. Therefore, for $i = 1, \ldots, m^*$, $\mathbf{E}[P^i]$ can be expressed as a function of $\mathbf{E}[P^1]$ as described in (54), since there is no instability caused by machines following the bottleneck. Inequality (48) for machine m^* does not hold, so

$$\begin{aligned}
\mathbf{E}[\alpha_0^{m^*}]C^{m^*} &< \mathbf{E}[d_0 + \sum_{i=m^*-1}^{1} (1-\alpha_0^i)P^i] \\
&= \mathbf{E}[d_0] + \sum_{i=1}^{m^*-1} \mathbf{E}[(1-\alpha_0^i)]\mathbf{E}[P^i] \\
&= \mathbf{E}[d_0] + \mathbf{E}[P^1] \sum_{i=1}^{m^*-1} \frac{\mathbf{E}[1-\alpha_0^i]}{\prod_{j=2}^{i} \mathbf{E}[\alpha_0^j]} \\
&< \mathbf{E}[d_0] \left(1 + \sum_{i=1}^{m^*-1} \frac{\mathbf{E}[1-\alpha_0^i]}{\prod_{j=1}^{i} \mathbf{E}[\alpha_0^j]} \right) \\
&= \mathbf{E}[d_0] \frac{1}{\prod_{j=1}^{m^*-1} \mathbf{E}[\alpha_0^j]},
\end{aligned} \tag{57}$$

showing that (46) does not hold for machine m^*. Thus, the equivalence between (46) and (48) is established and the result for single product follows.

QED

It remains to generalize the above to the multiple product situation. By using a class of policies that imposes *bounds on production quantities* it will be possible to provide a set of parameters that ensure no sharing of capacity when the system is operated in the PS mode.

To simplify the notation, assume we are dealing with a flow line constituted by $\hat{M}$ machines and with no re-entrance. In the PS mode, set $\hat{M} = KM$. Define Ω^m as the long run expected amount of work imposed on machine m by all products. This amount is given by

$$\Omega^m = \sum_{p=1}^{P} \frac{\mathbf{E}[d_0^p]}{\prod_{j=1}^{m} \mathbf{E}[\alpha_0^{jp}]} \quad \text{for all } m = 1, \ldots, \hat{M}. \tag{58}$$

Define the long run average load of machine m, for all $m = 1, \ldots, \hat{M}$, as

$$\Lambda^m = \frac{\Omega^m}{C^m}. \tag{59}$$

It is not difficult to see that it is necessary for all values of Λ^m to be below unity in order for the system to be stable.

Now, define as the long run bottleneck machine the one which has the highest long run average load. So, we have m^* as the machine for which

$$\Lambda^* = \frac{\Omega^{m^*}}{C^{m^*}} = \max_m \{\Lambda^m\}. \tag{60}$$

Define the share of each machine that can be used by each product in the long run as

$$C^{mp} = \frac{\mathbf{E}[d_0^p] / \prod_{j=1}^{m} \mathbf{E}[\alpha_0^{jp}]}{\Lambda^m}, \tag{61}$$

and set the values for Δ^{mp} that constitute the control variables for this problem as

$$\Delta^{m^+p} = \begin{cases} C^{m^*p} \prod_{j=m+1}^{m^*} \mathbf{E}[\alpha_0^{jp}] & \text{if } 1 \leq m \leq m^*, \\ C^{m^*p} & \text{if } m = m^*, \\ C^{m^*p} / \prod_{j=m^*+1}^{m} \mathbf{E}[\alpha_0^{jp}] & \text{if } \hat{M} > m \geq m^*. \end{cases} \tag{62}$$

Note that Δ^{m^+p} is the nominal inventory of product p that sits in front of machine m. That is why there is no need to define Δ^{1p}, which remains free as before[4]. The other control variables are the bounds on the input of new material per period for each product, which are

$$I^{\hat{M}^+p} = C^{m^*p} / \prod_{j=m^*+1}^{\hat{M}} \mathbf{E}[\alpha_0^{jp}]. \tag{63}$$

Given the fact that each value of $\Delta^{m^+p} \leq C^{mp}$, it is the case that, as long as $I_n^{m^+p} \leq \Delta^{m^+p}$, there is never a situation where the capacity of machine m has to be shared in the PS mode. This would always be the case if yield would be deterministic and exactly equal to its average value for all periods. Since in general $Pr(\alpha_n^{mp} > \mathbf{E}[\alpha_0^{mp}]) > 0$, we cannot ensure that the available inventory for all products sitting in front of a given machine is always such that its summation is below the machine's capacity. Thus, in the PS mode, there will be periods where sharing does indeed occur and equation (44) would have to be used explicitly to establish stability. It was said earlier that such dynamic equation is too cumbersome to be tackled. This implies that it is not possible to derive stability just by imposing a bound on the new material entering the system as it was done in Section 4. It is necessary to add further features to the control policies in order to obtain an instance that ensures no sharing in the PS mode and which can constitute an upper bound on the optimal cost, while maintaining stability.

[4] The same is true of the non negativity constraint.

The natural extension of class Π_1, furthers the extension proposed in Section 4 by adding a new set of variables. These new variables impose bounds on the amount of material allowed to enter production for each product at every machine on any given period. This way, one imposes a maximum share that each product can take from each machine, even if there is available inventory to produce more. This class of control policies, which will be called Π_2, turns out to be the sensible thing to do from the practitioners' point of view as well[5].

With this broader class of base stock policies in mind, the obvious instance which ensures stability and constitutes an upper bound on the cost of the optimal solution is such that all the new variables are equal to Δ^{m^+p} as well. That is, the additional bound for machine m to produce product p is the nominal value of the associated delta variable.

As was remarked at the end of Section 4, it may also be the case here that the optimal values for those bounds are such that sharing will eventually occur. It should be clear that there is no intention of running these systems as P independent production lines. Doing that would signify losing the flexibility allowed by the sharing of resources. For instance, it was discussed in [6] that the best performance in the TS mode was always better than the best performance in the PS mode. The greater the flexibility the better potential use one can make of the available resources. However, it may be the case that such flexibility may need a minimum amount of restraint to ensure *fairness* for all the products. Again, the use of the bounds is only essential to establish stability for infinite horizon systems.

Thus, the above discussion established the following theorem.

Theorem 12. Suppose the demand $\{d_n^p, -\infty < n < \infty\}$ is ergodic as well as stationary. Additionally, suppose the random yield $\{\alpha_n^{kmp}, -\infty < n < \infty\}$ is ergodic and stationary. If equation (45) holds, then the shortfall process is stable for multiple product systems operated in the PS mode, using class Π_2.

We argued in terms of a flow line composed of $\hat{M}$ machines. When a re-entrant system with K levels and M machines is operated in the PS mode it is transformed into a flow line with no re-entrance, where it is possible to map each pair (km) into a global ordering for $\hat{M} = K \times M$ machines.

Once the stability condition has been established, all the other results discussed in Section 3.1 are trivially derived. Theorem 4 and the associated corollary are the exceptions. In order to characterize the regeneration times we need one additional assumption, due to the presence of random yield. Additionally to condition (15), the following condition has to hold so that the shortfall process returns to the origin infinitely often, with probability one

$$Pr(\alpha^{kmp} = 1) > 0, \quad k = 1, \ldots, K\,; m = 1, \ldots, M\,; p = 1, \ldots, P. \tag{64}$$

If this does not hold, then the convergence of the shortfalls to zero can only occur in infinite time, since it will be accomplished through a geometric series.

[5] Clearly, $\Pi_0 \subset \Pi_1 \subset \Pi_2$.

5.2 Stability and Regeneration for Totally Shared Systems

The stability condition for the TS mode is the natural extension of the previous condition for random yield in the PS mode.

Theorem 13. Suppose the demand $\{d_n^p, -\infty < n < \infty\}$ is ergodic as well as stationary. Additionally, suppose the random yield $\{\alpha_n^{kmp}, -\infty < n < \infty\}$ is ergodic and stationary. If

$$\sum_{k=1}^{K}\sum_{p=1}^{P}\frac{\mathbf{E}[d_0^p]}{\prod_{q,r=1,1}^{q,r=k,m}\mathbf{E}[\alpha_0^{qrp}]} < C^m \quad \text{for } m = 1,\ldots,M, \tag{65}$$

then the shortfall process is stable for multiple product systems operated in the TS mode, using class Π_2.

Proof: To establish this result we only need to produce an instance of class Π_2, defined in the earlier subsection. The instantiated parameters of Π_2 follow the same reasoning just presented at the end of the previous subsection. That is, compute the average work on each machine; define the machine with the highest average load; determine the average share that each product at each level demands from the bottleneck machine; and use that share to determine the values of Δ^{kmp} and the values for the bounds on the production for all the levels, stages, and products. Given those, the system operated in the TS mode never shares capacity across products and levels. Also, every share allocated is never below the average work imposed. This implies that the P decoupled systems are all stable and the cost incurred by such control variables constitutes an upper bound on the performance of the optimal control variables.

Therefore, the optimal values of these same control variables will have to incur a lower cost and have to necessarily maintain stability. Also, the optimal values of the control variables may be such that sharing of capacity does indeed occur and the TS mode really allows a flexible use of all the available capacity as intended.

QED

Taking into account the discussion on the regeneration times made at the end of the previous subsection, all the results discussed for the TS mode in Section 3.2 carry through trivially for this setting.

5.2.1 Remarks on the Class Π_2

The class Π_2 of modified base stock policies constitutes a similar qualitative step from Π_1 as this latter constituted from Π_0. It may be the case that while optimizing relative to the base stock levels and production bounds the optimal values are such that no sharing really occurs either in the PS or the TS mode. This only means that such is the optimal thing to do and may have no direct relation with the fact that policies from Π_1 or Π_0 may induce instability.

According to the discussion of results in [5], the existence of production bounds other than the net capacity may be beneficial in terms of minimizing operational costs, independent of the stability issue.

Modeling the production system by means of a periodic review inventory control turns out to allow the definition of a broad class of policies that can incorporate non idling features in a very natural way. The lack of this feature was one of the drawbacks of other approaches, as queueing networks is one paradigmatic example.

Other modifications can be added to these policies, namely the need to impose upper bounds on the amount of inventory sitting at each buffer, which could be of advantage due to cost considerations and also to tackle the existence of machine failures. However, the modifications introduced to Π_0 to generate Π_2 are the minimal needed to establish stability.

Note also that, when controlling systems with random yield, deciding to produce the exact difference between a target value and the present value of inventory is known to be non-optimal. Other classes of policies would have to be proposed in order to eventually achieve better performances. Namely, inflating each current production decision by the reciprocal of the expected random yield would be a good candidate for a first approximation, although this is also known to be non-optimal. This type of generalizations are outside the scope of the present work and are only here referred to clarify that there is no substantial claim on the class Π_2 other than it may allow lower costs than Π_0, it ensures stability for the re-entrant systems addressed here, and even ensures stability for more complex re-entrant systems as mentioned in Section 4.

6 Non Uniform Loads and Random Yield

Given the discussion of the previous two sections, the stability results for this setting are

Theorem 14. Suppose the demand $\{d_n^p, -\infty < n < \infty\}$ is ergodic as well as stationary. Additionally, suppose the random yield $\{\alpha_n^{kmp}, -\infty < n < \infty\}$ is ergodic and stationary. If

$$\sum_{p=1}^{P} \tau^{kmp} \frac{\mathbf{E}[d_0^p]}{\prod_{q,r=1,1}^{q,r=k,m} \mathbf{E}[\alpha_0^{qrp}]} < C^{km} \quad \text{for} \begin{cases} m = 1, \ldots, M \\ k = 1, \ldots, K \end{cases} \tag{66}$$

then the shortfall process is stable for multiple product systems operated in the PS mode, using class Π_2.

Theorem 15. Suppose the demand $\{d_n^p, -\infty < n < \infty\}$ is ergodic as well as stationary. Additionally, suppose the random yield $\{\alpha_n^{kmp}, -\infty < n < \infty\}$ is ergodic and stationary. If

$$\sum_{k=1}^{K}\sum_{p=1}^{P}\tau^{kmp}\frac{\mathbf{E}[d_0^p]}{\prod_{q,r=1,1}^{q,r=k,m}\mathbf{E}[\alpha_0^{qrp}]} < C^m \quad \text{for } m = 1,\ldots,M, \tag{67}$$

then the shortfall process is stable for multiple product systems operated in the TS mode, using class Π_2.

7 Conclusions

We have established the conditions for stability on multiple product re-entrant flow lines for a wide range of settings. Our stability results cover systems with perfect and random yield, systems with uniform or non uniform loads, and systems with any type of processing flow. We proposed a class of control policies for which the necessary stability conditions are also sufficient. This class of control policies is a variant of the capacitated multi-echelon base stock policies, where each production decision has an explicit upper bound. Our stability discussion places the emphasis on determining stable policies rather than determining conditions under which a given policy induces stability. One of the elegances of the stability discussion is that it agrees with some of the insights produced by the experimental data of [6] and works concurrently with them. Therefore, the classes of policies that ensure stability provide an important contribution of this paper for future research.

Although some of the features of the richer policies are not particularly new nor unexpected, their study is still relatively insignificant. That has to do with the complexity of those policies in terms of their analytical evaluation. However, as long as a general tool like IPA can be used, their study becomes an easier task to undertake.

More than stressing the fact that the necessary stability conditions are also sufficient, we believe that the reasoning behind the arguments that led to this property are the most relevant contributions of the present work, that may span to other contexts, queueing networks being a possible example.

The main insight provided by the stability discussion, is the fact that there is a definite advantage in controlling production with idling policies. Even when backlogs are high, there should be some restraint on the amounts of new material entering the system and on the amounts of material allowed to move to the next operation. Much of the research in the past has concentrated on non-idling policies for intuitive reasons. This paper clearly challenges that intuition for non acyclic systems, multiple products, non uniform loads, and random yield. The modifications proposed to Π_0 are the minimal needed to establish sufficiency of the necessary stability condition.

Future research will have to address the validation of the IPA approach for these new classes of policies and investigate their potential in terms of performance.

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Appendix

This appendix includes some of the proofs on auxiliary results, skipped along the paper that were omitted then to avoid loosing sight of the essentials and because some of them are relatively trivial extensions of other results published.

Their inclusion here is intended at making this document the more self contained possible.

Proof of Lemma 2

Define

$$\begin{aligned}&\Phi_1 = \Phi,\\&\Phi_n(\boldsymbol{Y},\boldsymbol{D}_1,\ldots,\boldsymbol{D}_n) = \Phi_{n-1}\left(\Phi(\boldsymbol{Y},\boldsymbol{D}_1),\boldsymbol{D}_2,\ldots,\boldsymbol{D}_n\right),\end{aligned}$$

$$\begin{aligned}&\phi_1 = \phi,\\&\phi_n(\boldsymbol{Y},\boldsymbol{D}_1,\ldots,\boldsymbol{D}_n) = ||\Phi_n(\boldsymbol{Y},\boldsymbol{D}_1,\ldots,\boldsymbol{D}_n)|| = \phi_{n-1}\left(\Phi(\boldsymbol{Y},\boldsymbol{D}_1),\boldsymbol{D}_2,\ldots,\boldsymbol{D}_n\right),\end{aligned}$$

$n = 2,3,\ldots$, with Φ and ϕ as in Lemma 1. Then

$$||\boldsymbol{Y}_n|| = \phi_n(\boldsymbol{Y}_0,\boldsymbol{D}_0,\ldots,\boldsymbol{D}_{n-1}), \ \text{a.s.} \tag{68}$$

Each ϕ_n is increasing and continuous.

For integer i, define ${}^i\boldsymbol{Y}_0$ such that $||{}^i\boldsymbol{Y}_0|| = 0$ and

$$||{}^i\boldsymbol{Y}_n|| = \phi_n(\boldsymbol{0},\boldsymbol{D}_{i-n},\ldots,\boldsymbol{D}_{i-1}), \ \ n \geq 1. \tag{69}$$

That is, $||{}^i\boldsymbol{Y}_n||$ is the nth-period total shortfall for a process starting at the origin a time $i-n$. Therefore, if $||\boldsymbol{Y}_0|| = 0$, then $||{}^i\boldsymbol{Y}_n||$ has the distribution of $||\boldsymbol{Y}_n||$, due to the stationarity of $\{\boldsymbol{D}_n\}$. Moreover, since ϕ is increasing,

$$\begin{aligned}||{}^i\boldsymbol{Y}_{n+1}|| &= \phi_{n+1}(0,\boldsymbol{D}_{i-n-1},\ldots,\boldsymbol{D}_{i-1}) \\&= \phi_n(\phi(0,\boldsymbol{D}_{i-n-1}),\boldsymbol{D}_{i-n},\ldots,\boldsymbol{D}_{i-1}) \\&\geq \phi_n(0,\boldsymbol{D}_{i-n},\ldots,\boldsymbol{D}_{i-1}) \\&= ||{}^i\boldsymbol{Y}_n||.\end{aligned} \tag{70}$$

This means that, for each i, $||{}^i\boldsymbol{Y}_n||$ increases almost surely to a limit as $n \to \infty$. Denote this limit by $||\tilde{\boldsymbol{Y}}_i||$. Notice that

$$\begin{aligned}||{}^{i+1}\boldsymbol{Y}_n|| &= \phi\left(\Phi_{n-1}(\boldsymbol{0},\boldsymbol{D}_{i-n+1},\ldots,\boldsymbol{D}_{i-1}),\boldsymbol{D}_i\right) \\&= \phi({}^i\boldsymbol{Y}_{n-1},\boldsymbol{D}_i).\end{aligned} \tag{71}$$

Letting n increase and using the continuity of ϕ, we conclude that

$$||\tilde{\boldsymbol{Y}}_{i+1}|| = \phi(\tilde{\boldsymbol{Y}}_i, \boldsymbol{D}_i) \tag{72}$$

for all i. For the last assertion in the lemma, notice (as above) that $||{}^0\boldsymbol{Y}_n||$ has the same distribution as $||\boldsymbol{Y}_n||$ if $||\boldsymbol{Y}_0|| = 0$, so that if $\{||{}^0\boldsymbol{Y}_n||, n \geq 0\}$ increases almost surely to $||\tilde{\boldsymbol{Y}}_0||$, then the distribution of $\{||\boldsymbol{Y}_n||, n \geq 0\}$ increases to that of $||\tilde{\boldsymbol{Y}}_0||$.

QED

Proof of Theorem 1

The proof follows a reasoning similar to the one used in [13] to prove their Theorem 1 by using here equations (10) and (12).

For level K and stage M the total shortfall process $\{||\boldsymbol{Y}_n^{KM}||, n \geq 0\}$ follows a Lindley recursion, (10). It follows from Loynes' analysis of the single-server queue that if $\mathbf{E}[||\boldsymbol{D}_0||] < C^{KM}$ then $||\tilde{\boldsymbol{Y}}_0^{KM}|| < \infty$, a.s., whereas if $\mathbf{E}[||\boldsymbol{D}_0||] > C^{KM}$ then $||\tilde{\boldsymbol{Y}}_0^{KM}|| = \infty$, a.s..

The proof proceeds by induction on the levels and stages from (K,M) down to 11, assuming that (13) holds. Suppose $||\tilde{\boldsymbol{Y}}_0^{km}||$ is finite, a.s.. To show that the same must be true of $||\tilde{\boldsymbol{Y}}_0^{(km)^-}||$, we argue that if $||\tilde{\boldsymbol{Y}}_0^{(km)^-}|| = \infty$, then we would have $\mathbf{E}[||\boldsymbol{D}_0||] \geq C^{(km)^-}$. Observe, first, that if $||\tilde{\boldsymbol{Y}}_n^{(km)^-}|| = \infty$, then so is $||\tilde{\boldsymbol{Y}}_{n+1}^{(km)^-}||$. In other words, the event $\{||\tilde{\boldsymbol{Y}}_n^{(km)^-}|| = \infty\}$ is invariant under a shift in the time index and must therefore have probability zero or one (by the ergodicity of demands).

Now we use the random variables $||{}^i\boldsymbol{Y}_n||$ defined in Lemma 2. As shown there, $||{}^i\boldsymbol{Y}_{n+1}|| \geq ||{}^i\boldsymbol{Y}_n||$, a.s., for all n and i. Moreover, $||{}^i\boldsymbol{Y}_{n+1}||$ has the same distribution as $||{}^{i+1}\boldsymbol{Y}_{n+1}||$, so $\mathbf{E}[||{}^{i+1}\boldsymbol{Y}_{n+1}|| - ||{}^i\boldsymbol{Y}_n||] \geq 0$; this holds, in particular, for the $(km)^-$-th component:

$$\mathbf{E}[||{}^{i+1}\boldsymbol{Y}_{n+1}^{(km)^-}|| - ||{}^i\boldsymbol{Y}_n^{(km)^-}||] \geq 0. \tag{73}$$

From (71) we know that $||{}^{i+1}\boldsymbol{Y}_{n+1}|| = \phi({}^i\boldsymbol{Y}_n, \boldsymbol{D}_i)$. So, $||{}^{i+1}\boldsymbol{Y}_{n+1}^{(km)^-}|| - ||{}^i\boldsymbol{Y}_n^{(km)^-}||$ is the increase in the echelon-$(km)^-$ total shortfall due to demand D_i, and therefore cannot exceed $||\boldsymbol{D}_i||$. Thus,

$$||{}^{i+1}\boldsymbol{Y}_{n+1}^{(km)^-}|| - ||{}^i\boldsymbol{Y}_n^{(km)^-}|| \leq ||\boldsymbol{D}_i||, \text{ for all } n \geq 0. \tag{74}$$

If every C^{km} is infinite, then the conclusion of the Theorem is immediate; suppose then that some C^{km} is finite. Then $\mathbf{E}[||\boldsymbol{D}_i||] < \infty$, so a consequence of Fatou's lemma and (74) is

$$\mathbf{E}[\limsup_{n\to\infty} ||{}^{i+1}\boldsymbol{Y}_{n+1}^{(km)^-}|| - ||{}^i\boldsymbol{Y}_n^{(km)^-}||] \geq \limsup_{n\to\infty} \mathbf{E}[||{}^{i+1}\boldsymbol{Y}_{n+1}^{(km)^-}|| - ||{}^i\boldsymbol{Y}_n^{(km)^-}||], \tag{75}$$

and, by (73), this is non negative. Now if $||\tilde{\boldsymbol{Y}}_0^{(km)^-}||$ is infinite while $||\tilde{\boldsymbol{Y}}_0^{km}||$ is finite, then

$$\limsup_{n\to\infty}\{||^{i+1}\boldsymbol{Y}_{n+1}^{(km)^-}|| - ||^{i}\boldsymbol{Y}_{n}^{(km)^-}||\} = \limsup_{n\to\infty}\max\Big\{0, ||^{i}\boldsymbol{Y}_{n}^{(km)^-}|| + ||\boldsymbol{D}_i|| - C^{(km)^-},$$
$$\sum_{p=1}^{P}\Big({}^{i}Y_n^{kmp} + d_i^p - (z^{kmp} - z^{(km)^- p})\Big)^+\Big\} - ||^{i}\boldsymbol{Y}_{n}^{(km)^-}||$$
$$= ||\boldsymbol{D}_i|| - C^{(km)^-},$$

implying $\mathbf{E}[||\boldsymbol{D}_i||] - C^{(km)^-} \geq 0$. Thus, if in fact $\mathbf{E}[||\boldsymbol{D}_i||] < C^{(km)^-}$, then $||\tilde{\boldsymbol{Y}}_0^{(km)^-}||$ must be finite with probability one.

Conversely, suppose that $\mathbf{E}[||\boldsymbol{D}_0||] > C^{km}$ and let (k,m) be the earliest level and stage for which this holds. From (12) we see that $||\boldsymbol{Y}_{n+1}^{km}|| \geq ||Y_n^{km}|| + ||\boldsymbol{D}_n|| - C^{km}$, and similarly

$$||^{0}\boldsymbol{Y}_{n+1}^{km}|| \geq \sum_{r=1}^{n+1}(||\boldsymbol{D}_{-r}|| - C^{km}). \tag{76}$$

Hence, letting n increase to ∞,

$$||\tilde{\boldsymbol{Y}}_0^{km}|| \geq \limsup_{n\to\infty}\sum_{r=1}^{n+1}(||\boldsymbol{D}_{-r}|| - C^{km}), \tag{77}$$

and this is ∞ when $\mathbf{E}[||\boldsymbol{D}_0||] - C^{km} > 0$. For qr such that level q and stage r occur after level k and stage m, notice that

$$\begin{aligned}
\boldsymbol{Y}_n^{qr} &= \boldsymbol{Y}_n^{(qr)^+} - (\boldsymbol{z}^{(qr)^+} - \boldsymbol{z}^{qr}) + \boldsymbol{I}_n^{qr} \\
&= \boldsymbol{Y}_n^{((qr)^+)^+} - (\boldsymbol{z}^{((qr)^+)^+} - \boldsymbol{z}^{(qr)^+}) + \boldsymbol{I}_n^{(qr)^+} - (\boldsymbol{z}^{(qr)^+} - \boldsymbol{z}^{qr}) + \boldsymbol{I}_n^{qr} \\
&= \boldsymbol{Y}_n^{((qr)^+)^+} - (\boldsymbol{z}^{((qr)^+)^+} - \boldsymbol{z}^{qr}) + \boldsymbol{I}_n^{(qr)^+} + \boldsymbol{I}_n^{qr} \\
&\vdots \quad \vdots \quad \vdots \\
&= \boldsymbol{Y}_n^{km} - (\boldsymbol{z}^{km} - \boldsymbol{z}^{qr}) + \sum_{s,t=q,r}^{(km)^-} \boldsymbol{I}_n^{st}
\end{aligned}$$

for all n, which leads to

$$\begin{aligned}
||\boldsymbol{Y}_n^{qr}|| &= ||\boldsymbol{Y}_n^{km}|| - (||\boldsymbol{z}^{km}|| - ||\boldsymbol{z}^{qr}||) + \sum_{s,t=q,r}^{(km)^-} ||\boldsymbol{I}_n^{st}|| \\
&\geq ||\boldsymbol{Y}_n^{km}|| - (||\boldsymbol{z}^{km}|| - ||\boldsymbol{z}^{qr}||)
\end{aligned}$$

because $||\boldsymbol{I}_n^{st}|| \geq 0$ for all n, s, and t.

From this we can conclude that $||\tilde{\boldsymbol{Y}}_0^{qr}|| = \infty$ if $||\tilde{\boldsymbol{Y}}_0^{km}|| = \infty$.

QED

Proof of Theorem 2

The proof follows exactly the same reasoning as that of Theorem 2 in [13].

It suffices to show that for all $||\boldsymbol{Y}_0||$, the process $\{||\boldsymbol{Y}_n||, n \geq 0\}$ eventually coincides with a copy started at zero when both are driven by the same demands. Notice that $||\boldsymbol{Y}_n^{km}||$ is always at least as large as the corresponding component of a copy started at zero. Since $||\boldsymbol{Y}_n^{KM}||$ follows a Lindley recursion withe negative drift, it hits zero at a finite time N_{KM}. Subsequently, it coincides with the $(KM)-$th component of the process started at zero. Suppose now that for all $n \geq N_{km}$, $(||\boldsymbol{Y}_n^{km}||, \ldots, ||\boldsymbol{Y}_n^{KM}||)$ coincides with the corresponding components started at zero. We claim that for some almost-surely finite $N_{(km)^-} \geq N_{km}$,

$$||\boldsymbol{Y}^{(km)^-}_{N_{(km)^-}}|| = \max\left\{0, \sum_{p=1}^{P}\left(Y_n^{kmp} + d_n^p - (z^{kmp} - z^{(km)^-p})\right)^+\right\}; \tag{78}$$

this will provide the coupling time for $||\boldsymbol{Y}^{(km)^-}||$ since $||\boldsymbol{Y}^{km}||$ has already coupled. Suppose there is no such $N_{(km)^-}$. Then

$$||\boldsymbol{Y}^{(km)^-}_{n+1}|| = ||\boldsymbol{Y}^{(km)^-}_n|| + ||\boldsymbol{D}_n|| - C^{(km)^-} \tag{79}$$

for all $n \geq N_{km}$, implying that $\liminf_n ||\boldsymbol{Y}^{(km)^-}_n|| = -\infty$, since $\mathbf{E}[||\boldsymbol{D}_0||] < C^{(km)^-}$. This is impossible, because the shortfalls are always non negative, so (78) must indeed occur in finite time. Subsequently, $||\boldsymbol{Y}^{(km)^-}||$ coincides with the copy started at zero. We conclude by induction that there is an N_{11}, finite a.s., such that the entire vector $||\boldsymbol{Y}_n||$ couples with the initially zero process at time N_{11}. From this it follows that $||\boldsymbol{Y}_n|| \Rightarrow ||\tilde{\boldsymbol{Y}}_0||$ since $||\tilde{\boldsymbol{Y}}_0||$ is the limit in distribution when $||\boldsymbol{Y}_0|| = 0$.

Uniqueness follows. If $||\hat{\boldsymbol{Y}}_0||$ is stationary then $||\hat{\boldsymbol{Y}}_n||$ couples with $||\tilde{\boldsymbol{Y}}_n||$ in finite time, implying that they must have the same distribution.

QED

Proof of Theorem 4

If $\mathbf{E}[||\boldsymbol{D}_0||] < C^{km}$, then $P(||D_0|| < C^{11}) > 0$. Consequently, under the conditions of the theorem there exists an ε with $\varepsilon < \min_{k,m} C^{km}$ and $\varepsilon/P \leq \min_{k,m,p}(z^{kmp} - z^{(km)^-p})$ such that $\delta \stackrel{\Delta}{=} P(d_0^1 \leq \varepsilon/P, \ldots, d_0^P \leq \varepsilon/P) > 0$. Since $\boldsymbol{Y}$ has a finite stationary distribution, there exists a constant $b > 0$ such that the set $B_b \subseteq \mathrm{R}^{K\times M}$ defined by

$$B_b = \{(y^{11}, \ldots, y^{KM}) : 0 \leq y^{kmp} \leq b/P, k = 1, \ldots, K; m = 1, \ldots, M; p = 1, \ldots, P\} \tag{80}$$

is visited infinitely often by $\boldsymbol{Y}$. We will show that there exists an integer $r \geq 0$ and a real q such that

$$P_x(||\boldsymbol{Y}_r|| = 0) \geq q > 0 \quad \text{for all } x \in B_b, \tag{81}$$

from which it follows that $\boldsymbol{Y}$ visits 0 infinitely often.

If $d_0^p \leq \varepsilon/P$, then either $||\boldsymbol{Y}_1^{KM}|| = 0$ or $||\boldsymbol{Y}_1^{KM}|| \leq ||\boldsymbol{Y}_0^{KM}|| - (C^{KM} - \varepsilon)$. Thus, every time a demand for all products falls in $[0, \varepsilon/P]$, the echelon-KM shortfall is decreased by at least $C^{KM} - \varepsilon$, until it reaches zero. Starting in B_b, it takes at most $r_{KM} = \lceil b/(C^{KM} - \varepsilon) \rceil$ consecutive such demands to drive that shortfall to zero. Thus, with $q_{KM} = \delta^{r_{KM}}$, we have $P_x(||\boldsymbol{Y}_{r_{KM}}^{KM}|| = 0) \geq q_{KM}$ for all $x \in B_b$.

Suppose now that $||\boldsymbol{Y}_0^{(km)^+}||, \ldots, ||\boldsymbol{Y}_0^{KM}|| = 0$ for some (k,m) and that $\boldsymbol{Y}_0^{kmp}|| \leq b/P$, for all $p = 1, \ldots, P$. With probability at least δ^n, shortfalls $(km)^+, \ldots, (K,M)$ will remain at zero for the next n transitions. Moreover, for any n, if $||\boldsymbol{Y}_n^{(km)^+}|| = 0$ and $||\boldsymbol{Y}_n^{km}|| > 0$, then the inventory $I_n^{(km)^+ p}$ available for use by stage (k,m) is greater or equal to $(z^{(km)^+ p} - z^{kmp})$, for all $p = 1, \ldots, P$, being it the case that the inequality holds for at least one product, because of (14). Thus, if $||d_n^p|| \leq \varepsilon/P$, stage (k,m) cannot be constrained by inventory, and either $||\boldsymbol{Y}_{n+1}^{km}|| = 0$ or $||\boldsymbol{Y}_{n+1}^{km}|| \leq ||\boldsymbol{Y}_n^{km}|| - (C^{km} - \varepsilon)$. If we set $r_{km} = \lceil b/(C^{km} - \varepsilon) \rceil$ then, with probability at least $q_{km} = \delta^{r_{km}}$. $||\boldsymbol{Y}^{km}||$ is driven to zero in r_{km} steps. We conclude that with probability at least $q = q_{11} \cdots q_{KM}$, $||\boldsymbol{Y}_{r_{11} + \cdots + r_{KM}}|| = 0$ for any $\boldsymbol{Y}_0 \in B_b$.

QED