On the Scalability and Convergence of Simultaneous Parameter Identification and Synchronization of Dynamical Systems

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Synchronization of dynamical systems allows two systems to have identical state trajectories. This method consists of an appropriate unidirectional coupling from one system (drive) to the other (response). A method to adapt a response system such that synchronization is achieved was proposed by Chen and Lü in 2002. However, this method has two limitations: first, it does not scale well if the number of parameters is greater than the state dimension, and second, the parameters are not guaranteed to converge. An adaptation law addressing these two limitations is presented. The feasibility and advantages of the proposed method are illustrated by numerical simulations.

1. Introduction

One of the challenges in designing autonomous robots, particularly when we expect these robots to show intelligent behavior while interacting with the physical world, is the capability of perceiving and making predictions about the world. In this sense, system identification methods manifest themselves as a powerful tool to achieve this. System identification is a fundamental method in science, allowing the construction of models from observed data [1, 2]. These methods allow the use of these models not only to understand physical processes, but also to make predictions about their future behavior. The background for this paper is the problem of segmenting perception in physical robots into meaningful events in real time [3, 4], following a dynamical systems approach [5]. This approach consists of simultaneously learning a model of the robot interaction with the environment and deriving predictions about its short-term evolution. The robot

and the world are seen as a coupled dynamical system, where the former adapts and synchronizes with the latter. Event boundaries are detected once synchronization is lost [3]. We can model this problem using two dynamical systems, one representing the world (initially unknown to the robot), and the other representing the model of the world made by the robot: the problem consists in (1) adapting the latter to the former, by identifying it in real time, and (2) synchronizing the latter to the former, such that sudden deviations of synchronization can be employed to identify events [6, 7].

Consider two identical continuous-time dynamical systems, designated *drive* (D) and *response* (R). Under the framework described, the drive corresponds to the world, while the response corresponds to the model made by the robot. It is well known that the state evolution of each dynamical system, when taken separately, may differ radically for slightly different initial conditions, namely in the case of chaotic dynamical systems [8, 9]. However, in the presence of a unidirectional coupling from the drive to the response system, synchronization of their state trajectories is known to occur [10–12]. In this paper we limit the discussion to the simplest coupling scheme, in which the response system receives the full state vector from the drive. In this situation it is easy to design a controller that synchronizes both systems using feedback linearization.

Such synchronization assumes that both drive and response have the same dynamical model. This paper addresses the problem of achieving synchronization of a response system when the dynamical model of the drive is unknown. In particular, we target the problem of simultaneous adaptation and synchronization of a response system, given an unknown drive. Two assumptions are made: (1) the response dynamical model depends linearly on a parameter vector, and (2) there is a value for this vector that makes both systems identical. In 2002, Chen and Lü proposed a method to simultaneously adapt this parameter vector and to make both systems synchronized [13]. Lyapunov's second method was used to prove the feasibility of this method. However, due to the construction of the Lyapunov function employed, convergence of the response parameters is not guaranteed, as described at the end of Section 2. This has two consequences that prevent the general usage of this method. First, it does not scale in complexity: if the dimension of the parameter vector is greater than the dimension of the state vector, convergence is not guaranteed. And second, even with a small number of parameters, Chen's proof does not guarantee effective convergence of the parameters.

In this paper we address both of these problems, presenting a convergence proof for the simultaneous synchronization and adaptation of the response to an arbitrary drive system. Moreover, numerical sim-

ulations comparing the proposed approach with Chen's method illustrate the benefits of the approach.

Chaotic synchronization was first introduced by Pecora and Carrol in 1990 [14]. Since then, many publications have deepened our knowledge about this concept [11, 12, 15, 16]. A method for synchronizing the Rössler and Chen chaotic systems using active control was proposed by Agiza and Yassen [16]. However, the approach is specific to these particular systems. Chen and Lü proposed a method to perform simultaneous identification and synchronization of chaotic systems [13], but the results show some limitations, which are detailed in Section 2.

The paper is structured as follows: Section 2 formally states the problem, followed by the proposed solution in Section 3; experimental results are presented in Section 4, and Section 5 concludes the paper.

2. Problem Statement

Consider two dynamical systems, called drive and response, with a unidirectional coupling between them. Throughout this paper we will assume that both drive and response systems are identical, apart from a parameter vector, which is unknown. The goal of the adaptation law is to determine this parameter vector. Consider the drive system modeled by

$$\dot{x} = f(x) + F(x)\,\theta, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $\theta \in \mathbb{R}^m$ is a parameter vector. The nonlinear functions that support the model are $f : \mathbb{R}^n \to \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^{(n \times m)}$. The coupling between the drive and the response systems consists in a bias term, called *synchronization input*, from the drive to the response. The response system is identical to the drive, except for the parameter vector $\alpha \in \mathbb{R}^m$ and the synchronization input U,

$$\dot{y} = f(y) + F(y)\,\alpha + U(y, x, \alpha), \tag{2}$$

where $y(t) \in \mathbb{R}^n$ is the response state vector and $U : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the synchronization control function. This function U realizes the controller that, given the state input from the drive, synchronizes the response system.

Define the state error $e = y - x$ and the parameter error $\Delta = \alpha - \theta$; the simultaneous adaptation and synchronization problem consists in the design of a controller U and of a parameter adaptation law for α such that both $\lim_{t \to \infty} e(t) = 0$ and $\lim_{t \to \infty} \Delta(t) = 0$.

Chen proposes in [13] a solution to this problem in the form of an adaptation law for α.

Assumption 1. There is a controller U and a scalar function $V(e)$ that, for $\alpha = \theta$, satisfies both (*i*) $c_1 \|e\|^2 \le V(e) \le c_2 \|e\|^2$ and (*ii*) $\dot{V}(e) \le -W(e)$, for c_1, c_2 positive constants, $W(e)$ a positive definite function, and $U(x, x, \theta) = 0$.

For example, the controller

$$U(y, x, \theta) = -e + f(x) - f(y) + [F(x) - F(y)]\,\theta, \qquad (3)$$

and the function $V(e) = e^T e / 2$ satisfies this assumption.

Theorem 1. Under Assumption 1, the adaptation law

$$\dot{\alpha} = -F^T(x)[\text{grad } V(e)]^T, \text{ where grad } V(e) = \frac{\partial V}{\partial e}(e), \qquad (4)$$

stabilizes the system at the equilibrium point $e = 0$, $\alpha = \theta$.

Proof. See [13]. $\square$

In the proof of this theorem, Chen employs the Lyapunov function

$$V_1(e, \alpha) = \frac{1}{2}\,e^T e + \frac{1}{2}\Delta^T \Delta. \qquad (5)$$

There is a hidden assumption in the proof: it only holds if $U(y, x, \alpha) - U(y, x, \theta) = [F(x) - F(y)]\,\Delta$ (which is true if the controller in equation (3) is used).

Still, two problems remain that compromise the applicability of this result. The first one is that equation (4) does not guarantee a strict definite positiveness of $-\dot{V}_1$; in particular,

$$\dot{V}_1(0, \alpha) = 0 + \Delta^T \dot{\alpha} = 0 \qquad (6)$$

for any value of α, since $\dot{\alpha} = 0$ for e identically zero (equation (4)). Note that asymptotical stability in Lyapunov's second method requires positive definiteness of $-\dot{V}_1$ [17]. In practice, as the synchronization error e approaches zero, the magnitude of the parameter error Δ is not guaranteed to decrease. The second problem concerns the null space of $F^T(x)$: according to equation (4), the parameter vector α remains unchanged as long as grad $V(e)$ lies in the null space of $F^T(x)$. If $V(e) = e^T e / 2$ and the state error e lies in this null space, $e \in \text{Null}(F^T(x))$, then $\dot{\alpha} = -F^T(x)$, $e = 0$, that is, the parameter vector α remains unchanged, even if $\alpha \ne \theta$.

3. Proposed Solution

Let us first obtain a controller function U that achieves synchronization, assuming that the true value of the parameter vector is known, $\alpha = \theta$. In this situation, the error state e has the following dynamics:

$$\dot{e} = f(y) - f(x) + [F(y) - F(x)]\,\theta + U(y, x, \theta). \tag{7}$$

Considering now the positive definite Lyapunov function

$$V(e) = \frac{1}{2}\,e^T\,e, \tag{8}$$

its time derivative is $\dot{V} = e^T\,\dot{e}$. Taking the controller

$$U(y, x, \theta) = -K\,e - f(y) + f(x) - [F(y) - F(x)]\,\theta, \tag{9}$$

where K is an ($n \times n$) positive definite matrix, we have that $\dot{e} = -K\,e$. Matrix K is thus related with the synchronization rate. Since $-\dot{V} = e^T K\,e$ is a positive definite function, for a positive definite K, the system in equation (7) is globally uniformly asymptotically stable [17] at the equilibrium point $e = 0$. Note that this controller satisfies Chen's assumption referred to in Section 2.

Consider now the positive definite Lyapunov function

$$V(e, \Delta) = \frac{1}{2}\,e^T\,e + \frac{1}{2}\,\Delta^T\,\Delta. \tag{10}$$

This function is zero if and only if both the response is synchronized with the drive, and its parameters equal the parameters of the drive. The dynamics of the error e, while using the controller in equation (9), are then

$$\dot{e} = -K\,e + F(x)\,\Delta. \tag{11}$$

By left-multiplying this equation by $F^T(x)\,L$, where L is an ($n \times n$) positive definite matrix (which is related with the adaptation rate as discussed later), and transposing the result, the relation is found to be

$$\Delta^T\,F^T(x)\,L^T\,F(x) = \dot{e}^T\,L^T\,F(x) + e^T\,K^T\,L^T\,F(x). \tag{12}$$

We are now in the condition to prove the main result of this paper.

Theorem 2. Assuming that there is a constant matrix L such that $G(x) = F^T(x)\,L^T\,F(x)$ is positive definite for all x, the adaptation law

$$\dot{\alpha} = -F^T(x)\,[(L\,K + I)\,e + L\,\dot{e}], \tag{13}$$

together with the controller in equation (9), globally uniformly stabilizes both the error system in equation (11) at $e = 0$ and the parameter error at $\Delta = 0$.

Proof. Considering the Lyapunov function in equation (10), we have $\dot{V} = e^T \dot{e} + \Delta^T \dot{\Delta}$. Taking the adaptation law in equation (13) together with equation (12), while noting that $\dot{\Delta} = \dot{\alpha}$, the result is

$$\dot{V} = -e^T K e - \Delta^T G(x) \Delta. \tag{14}$$

Since $G(x)$ is assumed to be positive definite, $-\dot{V}$ is also positive definite, from which we can conclude that $(e, \Delta) = (0, 0)$ is a globally uniformly asymptotically stable [17] equilibrium point of equation (11). □

This theorem implies both synchronization $(y = x)$ and correct identification of the parameters $(\alpha = \theta)$. Note that the practical use of the proposed adaptation law in equation (13) requires knowledge of the error time derivative $\dot{e}$, which in principle can be obtained (or estimated) from the error evolution.

The choice of the constant matrices K and L has an impact on the convergence rate. If $\alpha = \theta$, the error system is $\dot{e} = -K e$, meaning that the error decreases asymptotically to zero according to first-order linear dynamics with a time constant determined by K. If $e = 0$, the parameter error has the dynamics $\dot{\Delta} = -F^T(x) L F(x) \Delta$, and thus the magnitude of L impacts the convergence rate of the parameters. Simple choices for K and L are diagonal matrices with constant values, $K = k I$ and $L = l I$, for two positive scalars k and l. Thus, the state and parameter error dynamics become $\dot{e} = -k e$ and $\dot{\Delta} = -l F^T(x) F(x) \Delta$.

Since $F(x)$ is an $(n \times m)$ matrix, its rank is lower or equal to $\min(n, m)$, and thus the rank of $G(x)$ is also lower and equal to $\min(n, m)$. However, in order for $G(x)$ to be positive definite, its rank has to be equal to m (the dimension of the parameter vector θ), and thus $n \geq m$ is a necessary condition for $G(x)$ to be full rank. This means that there is an upper bound to the amount of parameters m, in order for convergence to be guaranteed. This largely limits the flexibility of the response system to adapt to arbitrary drive systems, in particular with a large amount of parameters.

To tackle this problem we propose augmenting the $F(x)$ matrix with extra rows, as many as needed, in order for $G(x)$ to become full rank. First, let us designate by $x^*(t)$ a new state vector consisting in the concatenation of time-delayed versions of the original state vector $x(t)$,

$$x^* = [x_0 \, x_1 \, ... \, x_r]^T, \tag{15}$$

where $x_i(t) = x(t - i \delta)$, for $i = 1 ... r$ and $\delta > 0$. Using this state vector, the drive system becomes

$$\dot{x}^* = f^*(x^*) + F^*(x^*) \theta, \tag{16}$$

where

$$f^*(x^*) = \begin{bmatrix} f(x_0) \\ \vdots \\ f(x_r) \end{bmatrix} \quad \text{and} \quad F^*(x^*) = \begin{bmatrix} F(x_0) \\ \vdots \\ F(x_r) \end{bmatrix}. \tag{17}$$

This augmented system is equivalent to the one in equation (1), as the additional state dimension corresponds to time-delayed versions of the original system. The response system, with state vector $y^* \in \mathbb{R}^{(r+1)n}$, takes the form

$$\dot{y}^* = f^*(y^*) + F^*(y^*)\alpha + U^*(y^*, x^*, \alpha). \tag{18}$$

These two coupled systems in equation (16) and equation (18) with state vectors x^* and y^* can be viewed as a new pair of drive and response systems by themselves, with error vector $e^* = y^* - x^*$. Thus, the results obtained can be directly applied here: the synchronization controller becomes

$$\begin{aligned} U^*(y^*, x^*, \alpha) = \\ -K^* e^* - f^*(y^*) + f^*(x^*) - [F^*(y^*) - F^*(x^*)]\alpha, \end{aligned} \tag{19}$$

where the matrix K^* can be a $((r+1)n \times (r+1)n)$ block diagonal formed by K matrices,

$$K^* = \begin{bmatrix} K & 0 & 0 \\ \hline 0 & K & 0 \\ \hline 0 & 0 & \ddots \end{bmatrix}. \tag{20}$$

The adaptation law then becomes

$$\dot{\alpha} = -F^{*T}(x^*)[(L^* K^* + I)e^* + L^* \dot{e}^*], \tag{21}$$

where L^* is a $((r+1)n \times (r+1)n)$ matrix, which can also take the form of a block diagonal in the same fashion as K^* in equation (20),

$$L^* = \begin{bmatrix} L & 0 & 0 \\ \hline 0 & L & 0 \\ \hline 0 & 0 & \ddots \end{bmatrix}. \tag{22}$$

If both K^* and L^* have the block diagonal structure as in equation (20) and equation (22), the adaptation law in equation (21) can be simplified into

$$\dot{\alpha} = -\sum_{i=0}^{r} F^T(x_i)[(L K + I)e_i + \dot{e}_i], \tag{23}$$

where $e_i = y_i - x_i$ and $\dot{e}_i = \dot{y}_i - \dot{x}_i$.

With this augmented system, we can prove convergence when $n < m$ with Corollary 1.

Corollary 1. If matrix $G^*(x) = (F^*)(x)^T (L^*)^T F^*(x)$ is full rank for all x, then the response system in equation (18), together with the adaptation law in equation (21), globally uniformly stabilizes both the error system in equation (11) at $e = 0$ and the parameter error at $\Delta = 0$.

Proof. The equivalent drive and response systems in equation (16) and equation (18) satisfy the conditions of Theorem 2, as long as $G^*(x)$ is full rank. $\square$

The rank of $G^*(x)$ cannot be guaranteed *a priori,* but a necessary condition corollary can still be stated.

Corollary 2. If F has rank $n < m$, then $r \geq \lceil (m/n) - 1 \rceil$ is a necessary condition for G^* to be full rank.

Proof. The rank of $G^* = (F^*)^T (L^*)^T F^*$ is at most $\min[(r+1)n, m]$. Since G^* is an $m \times m$ matrix, in order to be full rank, $(r+1)n \geq m$ has to hold. Therefore, $r \geq (m/n) - 1$, but since r is an integer, its lower bound is $\lceil (m/n) - 1 \rceil$. $\square$

In general, as r is arbitrary, it can be expected that there is a value of r large enough that makes G^* full rank.

Comparing the obtained adaptation law in equation (23) with equation (13), it can be observed that the gradient of the parameters depends on several time-delayed samples of the error e (as well as on their derivatives $\dot{e}$). A possible insight into this result comes from the observation that, if $m > n$, the degrees of freedom of e are not enough to produce a meaningful gradient for α if the law in equation (13) is employed. However, with equation (23), which depends on e^* with $(r+1)n$ degrees of freedom, the gradient of α can have the full dimensionality of m.

4. Experimental Results

This section presents numerical results illustrating the theoretical results derived in Section 3. Two classical chaotic systems were used: the Lorenz oscillator [18], commonly used in the chaotic synchronization literature for numerical simulations [11–13], and the Rössler attractor, designed to behave similarly to the Lorenz system while being easier to understand [19]. Simultaneous identification and synchronization is simulated, while comparing the performance of Chen's method [13] with the one proposed here. For Chen's method we used the controller in equation (3) with the adaptation law in equation (4),

and for our method we used the controller in equation (19) with the adaptation law in equation (23).

The Lorenz oscillator is a three-dimensional dynamical system that behaves chaotically for a certain set of parameters [18]. In the form of equation (1), it can be written as

$$
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ -y - xz \\ xy \end{bmatrix} + \begin{bmatrix} y - x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & -z \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \tag{24}
$$

where x, y, and z are state variables and θ_1, θ_2, and θ_3 are system parameters. The Lorenz oscillator was synchronized with a response system, which is specified by four parameters. In the form of equation (2), it can be written as

$$
\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} 0 \\ -v - uw \\ uv \end{bmatrix} + \begin{bmatrix} v - u & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & -w & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{25}
$$

where u, v, and w are state variables and α_1, α_2, α_3, and α_4 are the parameters. Note that the rank of the $F(x)$ matrix in equation (25) is at most three, while the response system uses four parameters; α_4 is an unnecessary parameter that is not present in the drive in equation (24), being artificially introduced to compare the two approaches when $m > n$. As shown in Section 2, under these conditions Chen's method is not guaranteed to converge, while Corollary 2 requires $r \geq 1$ for G^* to be full rank, and thus a necessary condition for convergence (as Corollary 1).

For this simulation, the classical parameter values for the Lorenz system were used: $[\theta_1, \theta_2, \theta_3]^T = [10, 28, 8/3]^T$. The initial states of the drive system and the controlled system were arbitrarily set to $[8, 9, 10]^T$ and $[3, 4, 5]^T$, respectively. The parameters of the response system had zero initial condition. The L and K parameters were set to $10\,I$ and $0.1\,I$.

Figure 1 shows the numerical results of parameter identification for parameter α_4. (All simulations were performed using Python together with SciPy [20] and PyDDE [21] libraries.) Note the trend for the parameter convergence to be faster for higher values of r. Figure 2 shows the results of parameter identification for the parameter α_4 for Chen's method over a longer time horizon. While Chen's method is not able to identify this parameter even after 1000 seconds, our method allows for a significantly faster convergence (under 200 sec-

onds). Figure 3 shows the results of parameter identification for α_3. Table 1 shows the time it takes for the parameter identification error to fall below a percentage of the real parameter value. Note again that the convergence is faster for higher values of r. It is interesting to note that, for instance, during the last 20 seconds of the simulation, the coefficient of variation (defined as the ratio $\sigma / |\mu|$, where σ is the standard deviation and μ the sample mean) of the root mean square error is of 4.42×10^{-3} for Chen's method, while for our method it is of 1.21×10^{-4} ($r = 3$) and 1.36×10^{-6} ($r = 5$). Chen's method is not able to correctly identify this parameter, with its value oscillating around the true value of θ_3. Our method, however, allows for a lower variance in the parameter identification. Figure 4 shows the synchronization error as measured by the Lyapunov function in equation (8). Both Chen's method and ours are able to drive the synchronization error to zero. Our method, however, shows near-instantaneous convergence. Also, the magnitude of the error is reduced by comparison to Chen's method.
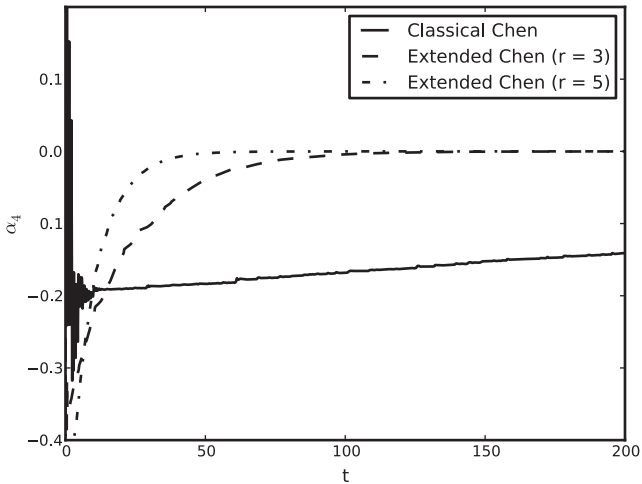

Figure 1. Lorenz system: graph of parameter identification results for α_4. Solid line: Chen's method in equation (4); dotted and dash-dotted lines: proposed method in equation (23) for $r = 3$ and $r = 5$.
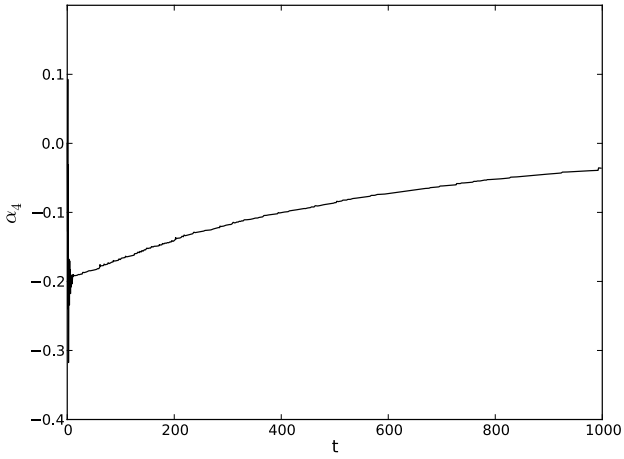
Figure 2. Lorenz system: graph of parameter identification results for α_4 using Chen's method in equation (4).
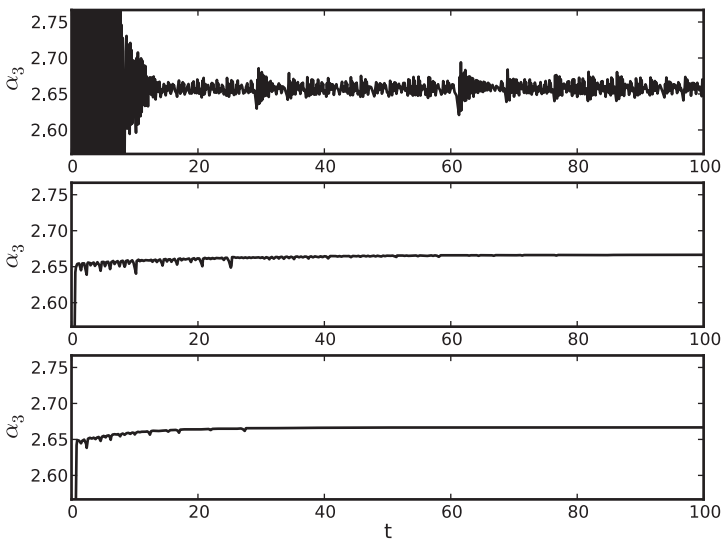

Figure 3. Lorenz system: plot of parameter identification results for α_3. Top plot: Chen's method in equation (4); middle and bottom plots: proposed method in equation (23) for $r = 3$ and $r = 5$.
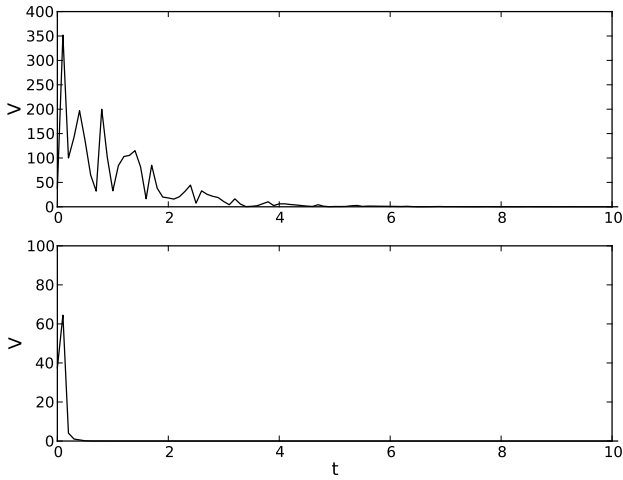
Figure 4. Lorenz system: graph of the Lyapunov function for synchronization error. Top plot: Chen's method in equation (4); bottom plot: proposed method in equation (23) for $r = 3$.

		Identification Error			
Method		1%	0.1%	0.01%	0.001%
Classical Chen		203.0	1609.8	3026.9	4640.3
Extended Chen	r=1	0.4	261.5	612.1	942.7
	r=2	10.3	104.8	226.7	308.5
	r=3	2.5	58.4	106.5	163.8
	r=4	2.5	31.1	70.9	94.8
	r=5	2.5	21.6	43.5	67.3

Table 1. Time to reach identification error ranges for parameter α_3 (in simulation seconds).

Similar results were obtained using the Rössler attractor. In the form of equation (1), it can be written as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -y - z \\ x \\ x\,z \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ y & 0 & 0 \\ 0 & 1 & -z \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \tag{26}$$

where x, y, and z are state variables and θ_1, θ_2, and θ_3 are system parameters. The Rössler system was synchronized with a response system specified by four parameters. In the form of equation (2), it can be written as

$$
\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} -v-w \\ u \\ u\,w \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ v & 0 & 0 & 1 \\ 0 & 1 & -w & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{27}
$$

where u, v, and w are state variables and α_1, α_2, α_3, and α_4 are the parameters. Again, the rank of the $F(x)$ matrix is three, while the number of parameters is four.

For this simulation, the commonly used parameter values for the Rössler system were used: $[\theta_1, \theta_2, \theta_3]^T = [0.1, 0.1, 14]^T$. The initial states of the drive system and the controlled system were arbitrarily set to $[8, 9, 10]^T$ and $[3, 4, 5]^T$, respectively. The parameters of the response system had zero initial condition. The L and K parameters were set to $10\,I$ and $0.1\,I$.

The improved convergence performance of the proposed method over Chen's is clearly visible in Figure 5, while parameter convergence is faster for higher values of r. Figure 6 shows the results of parameter identification for α_1, which, together with α_4, specifies the evolution of the state variable v. During the last 20 seconds of the experiment, the coefficient of variation of the root mean square error is of 2.96×10^{-2} for Chen's method, while for our method it is of 6.11×10^{-5} ($r = 3$) and 3.32×10^{-7} ($r = 5$). Chen's method cannot identify this parameter correctly, with its value oscillating around the true value of θ_1. On the other hand, our method allows for stable parameter identification. Again, convergence is faster for greater values of r. Finally, Figure 7 shows the synchronization error, as measured by the Lyapunov function in equation (8). Both methods drive the synchronization error to zero, while our method shows a significantly faster convergence. Also, the magnitude of the error is reduced by comparison to Chen's method.
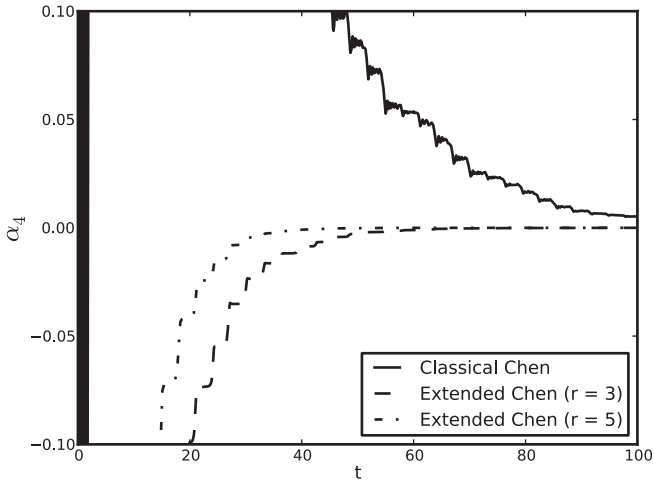
Figure 5. Rössler system: graph of parameter identification results for α_4. Solid line: classical Chen; dotted line: extended Chen ($r = 3$); dash-dotted line: extended Chen ($r = 5$).
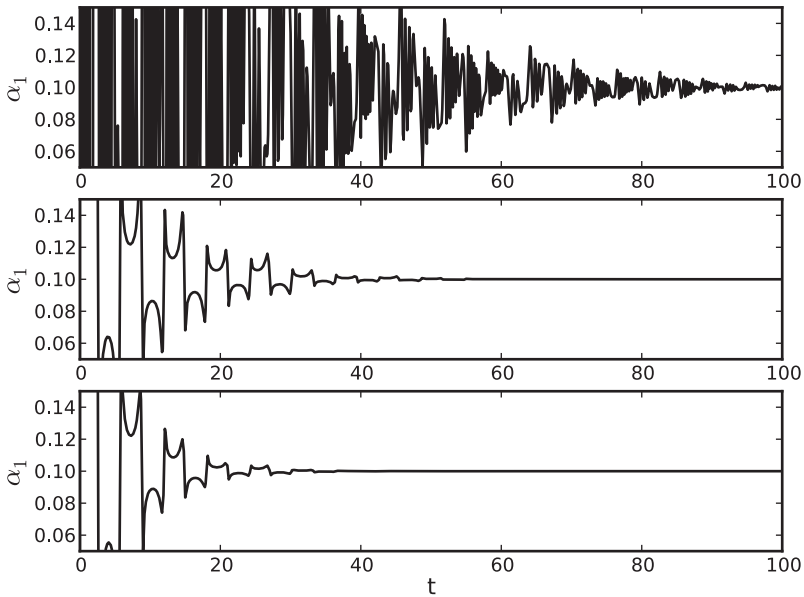

Figure 6. Rössler system: plot of parameter identification results for α_1. Top plot: classical Chen; middle plot: extended Chen ($r = 3$); bottom plot: extended Chen ($r = 5$).
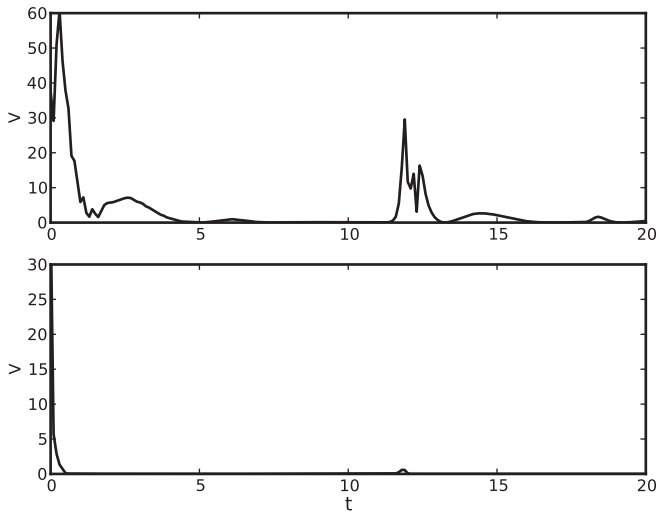
Figure 7. Rössler system: graph of the Lyapunov function for synchronization error. Top plot: classical Chen; bottom plot: extended Chen ($r = 3$).

5. Conclusions and Future Work

Building upon previous work in simultaneous parameters identification and synchronization of dynamical systems, this paper proposes an improved method that addresses limitations of the previously published Chen's method [13]. The proposed method is capable of handling arbitrarily large parameter space dimensions. Convergence proof of the method is provided, using Lyapunov's second method. Numerical results illustrate the proposed method, comparing it to Chen's and showing better performance in terms of both faster and less noisy parameter identification.

As future work, we consider the use of the unified chaotic system [22, 23], which is double-scroll, as a template for the response system. This could open the door for response systems capable of adapting and synchronizing to multi-scroll systems. We also contemplate the extension of the proposed method to complex dynamical networks [24, 25].

Acknowledgments

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