Stability of a Nonlinear Attitude Observer on $\mathrm{SO}(3)$ with Nonideal Angular Velocity Measurements

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Abstract— **This paper presents a nonlinear observer for attitude estimation on $\mathrm{SO}(3)$ and studies the stability properties of the system in the presence of nonideal inertial sensor measurements. Exploiting vector observations and biased angular velocity readings in the feedback law, almost global asymptotic stability of the origin is obtained and exponential convergence is guaranteed for an explicit region in the state space. Sufficient conditions in the observer design are proposed to yield exponential stability of the origin given worst-case initial alignment errors. Stability of the observer in the presence of angular measurement noise is obtained, and convergence to a desired neighborhood of the origin, for any initial condition in a known region, can be guaranteed by properly defining the observer parameters. The properties of the observer are illustrated in simulation for inertial sensor characteristics and initial alignment errors commonly found in practical setups.**

I. INTRODUCTION

Attitude estimation is a classical problem with a rich and fascinating history still holding a forefront position as the subject of intensive research [1]. Recent publications [2], [3], [4] provide important guidelines for the design of attitude observers, by pointing out topological issues that hinder global stabilization on non-Euclidean spaces such as the special orthogonal group $\mathrm{SO}(3)$.

Several nonlinear attitude observers and compensators have been proposed in recent literature [5], [6], [7]. Attitude observers based only on the rotation kinematics are of special interest for applications using inertial sensors and attitude aiding devices, such as rate gyros and vector readings, respectively [8], [9], [10], [11].

The present work proposes a nonlinear attitude observer, defined on $\mathrm{SO}(3)$, that yields an almost globally asymptotically stable (aGAS) equilibrium point at the origin, and exponential convergence of the estimation error within an explicit region. The stability properties are obtained for the case of biased angular velocity readings, and are a function of the design parameters. These can be determined in order to yield uniform exponential stability of the origin given worst-case initial estimation errors. The observer stability is also analyzed in the presence of noise in the angular velocity measurements, providing ultimate bounds for the attitude estimation error. Sufficient conditions to drive the attitude error to a desired neighborhood of the origin are given, which can be adopted in practical applications to satisfy accuracy specifications.

The observer extends the architecture proposed in [11], and builds on the derivation method presented in [12], which used a conveniently defined Lyapunov function to yield an output feedback law that exploits the sensor readings directly. The design parameters of the proposed observer are used to define properties such as the region of exponential convergence, the location of the unstable equilibria, and the ultimate bounds in the presence of noise, and hence are of interest in the implementation of the observer.

The paper is organized as follows. In Section II, the attitude estimation problem is described. Section III introduces the Lyapunov function used in the observer synthesis. In Section IV, the attitude observer is proposed and the stability properties in the presence of biased angular velocity readings are derived. Almost global asymptotic stabilization and exponential convergence of the estimation errors are demonstrated. In Section V, the stability results for the attitude observer with noise in the velocity measurements are presented. In Section VI, simulation results illustrate the stability properties of the observer. Section VII presents concluding remarks and comments on future work.

NOMENCLATURE

The notation adopted is fairly standard. The set of $n \times m$ matrices with real entries is denoted as $\mathrm{M}(n,m)$ and $\mathrm{M}(n) := \mathrm{M}(n,n)$. The sets of skew-symmetric, orthogonal, and special orthogonal matrices are respectively denoted by $\mathrm{K}(n) := \{\mathbf{K} \in \mathrm{M}(n) : \mathbf{K} = -\mathbf{K}'\}$, $\mathrm{O}(n) := \{\mathbf{U} \in \mathrm{M}(n) : \mathbf{U}'\mathbf{U} = \mathbf{I}\}$, $\mathrm{SO}(n) := \{\mathbf{R} \in \mathrm{O}(n) : \det(\mathbf{R}) = 1\}$, and the n-dimensional sphere and ball are described by $\mathrm{S}(n) := \{\mathbf{x} \in \mathbb{R}^{n+1} : \mathbf{x}'\mathbf{x} = 1\}$ and $\mathrm{B}(n) := \{\mathbf{x} \in \mathbb{R}^{n} : \mathbf{x}'\mathbf{x} \leq 1\}$, respectively. The minimum and maximum singular values of a matrix $\mathbf{A} \in \mathrm{M}(n,m)$ are denoted by $\sigma_{\min}(\mathbf{A})$ and $\sigma_{\max}(\mathbf{A})$, respectively.

II. PROBLEM FORMULATION

In this section, the concepts involved in the attitude estimation problem are introduced. The rigid body kinematics are described by

$$\dot{\mathcal{R}} = \mathcal{R}(\boldsymbol{\omega})_{\times},$$

where $\mathcal{R}$ is the shorthand notation for the rotation matrix ${}^{L}_{B}\mathbf{R}$ from body frame $\{B\}$ to local frame $\{L\}$ coordinates, $\boldsymbol{\omega}$ is the body angular velocity expressed in $\{B\}$, and $(\mathbf{a})_{\times}$

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is the skew symmetric matrix defined by the vector $\mathbf{a} \in \mathbb{R}^3$ such that $(\mathbf{a})_\times \mathbf{b} = \mathbf{a} \times \mathbf{b}$, $\mathbf{b} \in \mathbb{R}^3$.

The body angular velocity measurement, denoted by $\boldsymbol{\omega}_r$, is obtained by a rate gyro sensor triad. Vector observations are a function of the rigid body's orientation. On-board sensors such as magnetometers, star trackers and pendulums, among others, provide vector observations expressed in body frame coordinates

$$\mathbf{h}_{r\,i} = {}^B\mathbf{h}_i := \mathcal{R}'\,{}^L\mathbf{h}_i, \tag{1}$$

where $i = 1..n$ is the vector index, n is the number of vector measuring sensors, and the vector representation in the local coordinate frame $\{L\}$, denoted by ${}^L\mathbf{h}_i$, is known.

The proposed observer estimates the orientation of the rigid body by computing the kinematics

$$\dot{\hat{\mathcal{R}}} = \hat{\mathcal{R}}(\hat{\boldsymbol{\omega}})_\times,$$

where $\hat{\mathcal{R}}$ is the estimated attitude and $\hat{\boldsymbol{\omega}}$ is the feedback term constructed to compensate for the attitude estimation error.

The attitude error is defined as $\tilde{\mathcal{R}} := \hat{\mathcal{R}}'\mathcal{R}$, and the Euler angle-axis parametrization of $\tilde{\mathcal{R}}$ is described by the rotation vector $\boldsymbol{\phi} \in \mathrm{S}(2)$ and by the rotation angle $\varphi \in \begin{bmatrix}0 & \pi\end{bmatrix}$, yielding the DCM formulation [13], denoted by $\tilde{\mathcal{R}} = \mathrm{rot}(\varphi, \boldsymbol{\phi}) := \cos(\varphi)\mathbf{I} + \sin(\varphi)(\boldsymbol{\phi})_\times + (1 - \cos(\varphi))\boldsymbol{\phi}\boldsymbol{\phi}'$. The attitude error kinematics are a function of the angular velocity estimates and given by $\dot{\tilde{\mathcal{R}}} = -\tilde{\mathcal{R}}(\tilde{\mathcal{R}}'\hat{\boldsymbol{\omega}} - \boldsymbol{\omega})_\times$.

The objective of the present work is to define an attitude feedback law $\hat{\boldsymbol{\omega}}$ as function of the velocity readings $\boldsymbol{\omega}_r$ and vector observations (1), so that the closed loop attitude observer has an asymptotically stable equilibrium point at the origin ($\tilde{\mathcal{R}} = \mathbf{I}$) with the largest region of attraction possible.

III. SYNTHESIS LYAPUNOV FUNCTION

The attitude feedback law is derived resorting to the Lyapunov's stability theory and to a conveniently defined transformation of the vector observations, described in this section. Define the linear combination of the sensed vectors ${}^L\mathbf{h}_i$ expressed in the local coordinate frame as

$$ {}^L\mathbf{u}_j := \sum_{i=1}^{n} a_{ij}\,{}^L\mathbf{h}_i, \quad j = 1..n. \tag{2}$$

The vector transformation (2) is represented in matrix form by $\mathbf{U}_H = \mathbf{H}\mathbf{A}_H$, where $\mathbf{U}_H = \begin{bmatrix}{}^L\mathbf{u}_1 & \dots & {}^L\mathbf{u}_n\end{bmatrix}$, $\mathbf{H} = \begin{bmatrix}{}^L\mathbf{h}_1 & \dots & {}^L\mathbf{h}_n\end{bmatrix}$, $\mathbf{U}_H, \mathbf{H} \in \mathrm{M}(3, n)$ and $\mathbf{A}_H = [a_{ij}] \in \mathrm{M}(n)$ is invertible by construction.

Let ${}^B\mathbf{u}_i = \mathcal{R}'\,{}^L\mathbf{u}_i$ and ${}^B\hat{\mathbf{u}}_i = \hat{\mathcal{R}}'\,{}^L\mathbf{u}_i$ be the nominal and the estimated representations of ${}^L\mathbf{u}_j$ in body frame coordinates, respectively. The corresponding matrix representation is ${}^B\mathbf{U}_H = \mathcal{R}'\mathbf{U}_H$, ${}^B\hat{\mathbf{U}}_H = \hat{\mathcal{R}}'\mathbf{U}_H$, where ${}^B\mathbf{U}_H = \begin{bmatrix}{}^B\mathbf{u}_1 & \dots & {}^B\mathbf{u}_n\end{bmatrix}$ and ${}^B\hat{\mathbf{U}}_H = \begin{bmatrix}{}^B\hat{\mathbf{u}}_1 & \dots & {}^B\hat{\mathbf{u}}_n\end{bmatrix}$, ${}^B\mathbf{U}_H, {}^B\hat{\mathbf{U}}_H \in \mathrm{M}(3, n)$.

The Lyapunov function is defined by the weighted estimation error of the transformed vectors,

$$V_H = \frac{1}{2}\sum_{i=1}^{n}({}^B\hat{\mathbf{u}}_i - {}^B\mathbf{u}_i)'\mathbf{W}({}^B\hat{\mathbf{u}}_i - {}^B\mathbf{u}_i)$$

$$= \frac{1}{2}\,\mathrm{tr}\left[(\mathbf{I} - \tilde{\mathcal{R}})\mathbf{W}(\mathbf{I} - \tilde{\mathcal{R}})'\mathbf{U}_H\mathbf{U}_H'\right], \tag{3}$$

where $\mathbf{W} \in \mathrm{M}(3)$ is a positive definite matrix, i.e. $\mathbf{W} > 0$, to be determined in the design of the observer.

The geometric configuration of the measured vectors is required to satisfy the following assumption, which is necessary and sufficient for attitude estimation, as discussed in [11] and references therein.

Assumption 1: There are at least two linearly independent vectors ${}^L\mathbf{h}_i$, that is, $\mathrm{rank}(\mathbf{H}) \geq 2$.

Using Assumption 1, necessary and sufficient conditions such that $V_H > 0$ are obtained.

Lemma 1: The Lyapunov function V_H has a unique global minimum if and only if Assumption 1 is verified.

Proof: Since $\mathbf{A}_H$ is invertible, Assumption 1 is equivalent to the linear independence of at least two ${}^L\mathbf{u}_i$. Given that $\mathbf{W} > 0$, then $V_H = 0$ if and only if $\sum_{i=1}^{n}\|{}^B\mathbf{u}_i - {}^B\hat{\mathbf{u}}_i\|^2 = 0$. Using [12, Lemma 1], the latter condition has an unique solution ($\tilde{\mathcal{R}} = \mathbf{I}$) if and only if two ${}^B\mathbf{u}_i$ are linearly independent, which concludes the proof. ∎

The transformation $\mathbf{A}_H$ considered in this work is described in the following proposition.

Proposition 2: Under Assumption 1 there is a nonsingular $\mathbf{A}_H \in \mathrm{M}(n)$ such that $\mathbf{U}_H\mathbf{U}_H' = \mathbf{I}$.

Proof: If $\mathrm{rank}\,\mathbf{H} = 3$, the derivation of $\mathbf{A}_H$ can be found in [11, Proposition 3]. If $\mathrm{rank}\,\mathbf{H} = 2$, the conditions of Proposition 2 can be satisfied by formulating an augmented attitude observer, for more details see [11, Appendix A]. ∎

Using the transformation $\mathbf{A}_H$ defined in Proposition 2, the Lyapunov function (3) is expressed by

$$V_H = \mathrm{tr}\left[(\mathbf{I} - \tilde{\mathcal{R}})\mathbf{W}\right] = \frac{1}{4}\|\mathbf{I} - \tilde{\mathcal{R}}\|^2\boldsymbol{\phi}'\mathbf{P}\boldsymbol{\phi} \tag{4}$$

where $\mathbf{P} = \mathrm{tr}(\mathbf{W})\mathbf{I} - \mathbf{W}$, $\mathbf{P} \in \mathrm{M}(3)$. In this work, the convergence properties of the observer are studied given the directionality of $\mathbf{P}$, that is an observer design parameter.

IV. OBSERVER SYNTHESIS AND STABILITY ANALYSIS

This section derives an observer for attitude estimation, in the presence of biased velocity measurements. The angular velocity reading is given by

$$\boldsymbol{\omega}_r = \boldsymbol{\omega} + \mathbf{b}_\omega.$$

The proposed synthesis Lyapunov function is augmented with a bias compensation error term

$$V = V_H + \frac{\gamma_b}{2}\|\tilde{\mathbf{b}}_\omega\|^2, \tag{5}$$

where $\tilde{\mathbf{b}}_\omega = \hat{\mathbf{b}}_\omega - \mathbf{b}_\omega$ is the bias compensation error. The time derivative of the Lyapunov function along the system trajectories is given by

$$\dot{V} = \dot{V}_H + \gamma_b\tilde{\mathbf{b}}_\omega\dot{\tilde{\mathbf{b}}}_\omega, \quad \dot{V}_H = -\mathbf{s}_{\tilde{\mathcal{R}}}'(\hat{\boldsymbol{\omega}} - \tilde{\mathcal{R}}\boldsymbol{\omega}),$$

where $\mathbf{s}_\mathcal{R} = (\tilde{\mathcal{R}}\mathbf{W} - \mathbf{W}\tilde{\mathcal{R}}')_\otimes$ and $(\cdot)_\otimes$ is the unskew operator such that $((\mathbf{w})_\times)_\otimes = \mathbf{w}$, $\mathbf{w} \in \mathbb{R}^3$. The feedback law for the angular velocity is defined as

$$\hat{\boldsymbol{\omega}} = \tilde{\mathcal{R}}(\boldsymbol{\omega}_r - \hat{\mathbf{b}}_\omega) + k_\omega \mathbf{s}_\mathcal{R} = \tilde{\mathcal{R}}(\boldsymbol{\omega} - \tilde{\mathbf{b}}_\omega) + k_\omega \mathbf{s}_\mathcal{R}, \quad (6)$$

so as to obtain a negative semi-definite derivative for the Lyapunov function. The resulting expression for the time derivative of the Lyapunov function is described by $\dot{V} = -k_\omega \mathbf{s}_\mathcal{R}'\mathbf{s}_\mathcal{R} + \tilde{\mathbf{b}}_\omega'(\gamma_b \dot{\tilde{\mathbf{b}}}_\omega + \tilde{\mathcal{R}}'\mathbf{s}_\mathcal{R})$. The nominal bias is considered constant, $\dot{\mathbf{b}}_\omega = \mathbf{0}$, hence $\dot{\hat{\mathbf{b}}}_\omega = \dot{\tilde{\mathbf{b}}}_\omega$, and the bias feedback law is defined as

$$\dot{\hat{\mathbf{b}}}_\omega = -k_{b_\omega}\mathbf{s}_\mathcal{R}, \quad (7)$$

and $\gamma_b = k_{b_\omega}^{-1}$ where k_{b_ω} is a positive scalar. The closed loop kinematics are given by

$$\dot{\tilde{\mathcal{R}}} = k_\omega \tilde{\mathcal{R}}(\tilde{\mathcal{R}}'\mathbf{W} - \mathbf{W}\tilde{\mathcal{R}}) + \tilde{\mathcal{R}}(\tilde{\mathbf{b}}_\omega)_\times, \quad (8a)$$

$$\dot{\tilde{\mathbf{b}}}_\omega = k_{b_\omega}(\tilde{\mathcal{R}}'\mathbf{W} - \mathbf{W}\tilde{\mathcal{R}})_\otimes, \quad (8b)$$

and the time derivative of the Lyapunov function is described by $\dot{V} = -k_\omega \mathbf{s}_\mathcal{R}'\mathbf{s}_\mathcal{R} \leq 0$. The set of points where $\dot{V} = 0$ are characterized in the following result.

Lemma 3 ([12]): Under Assumption 1, the set of points where $\dot{V} = 0$ is given by

$$C_\mathcal{R} = \{(\tilde{\mathcal{R}}, \tilde{\mathbf{b}}_\omega) \in \mathrm{SO}(3) \times \mathbb{R}^3 : \tilde{\mathcal{R}} = \mathbf{I} \vee \tilde{\mathcal{R}} = \mathrm{rot}(\pi, \boldsymbol{\phi} \in \mathrm{eigvec}(\mathbf{P}))\}.$$

The multiple equilibrium points contained in the set $C_\mathcal{R}$ illustrate the topological obstacles to continuous state feedback on manifolds, discussed in [2], [4], [14]. However, $C_\mathcal{R}$ is of zero measure and it is possible to show that, for almost all initial conditions, the solutions of the system (8) are attracted to the origin. For the case of unbiased velocity measurements, aGAS and exponential stability of $\tilde{\mathcal{R}} = \mathbf{I}$ are obtained.

Theorem 4 ([12]): Consider the system (8a) with $\tilde{\mathbf{b}}_\omega = 0$. The attitude error $\tilde{\mathcal{R}} = \mathbf{I}$ is exponentially stable with region of attraction given by $R_A = \{\tilde{\mathcal{R}} \in \mathrm{SO}(3) : \tilde{\mathcal{R}} = \mathrm{rot}(\varphi, \boldsymbol{\phi}), \boldsymbol{\phi} \in \mathrm{S}(2), |\varphi| < \pi\}$. For any initial condition in the region of attraction, the trajectories satisfy

$$\|\tilde{\mathcal{R}}(t) - \mathbf{I}\| \leq \|\tilde{\mathcal{R}}(t_0) - \mathbf{I}\| e^{-k_\omega(1+\cos(\varphi(t_0)))\sigma_{\min}(\mathbf{P})(t-t_0)}.$$

For biased angular velocity measurements, the stability of the observer (8) is derived by showing *i)* exponential stability of the origin given bounded initial estimation errors, and *ii)* aGAS of the origin. The combination of these properties yields that the origin is aGAS, with exponential convergence of the trajectories in a known region.

A. Exponential Stability

The exponential stability of the origin is obtained by formulating sufficient conditions that exclude the set of points $\tilde{\mathcal{R}} = \mathrm{rot}(\pi, \boldsymbol{\phi})$.

Lemma 5: The attitude and bias estimation errors, $\tilde{\mathcal{R}}$ and $\tilde{\mathbf{b}}_\omega$ respectively, are bounded. For any initial condition such that

$$k_{b_\omega} > \frac{\|\tilde{\mathbf{b}}_\omega(t_0)\|^2}{2(2\sigma_{\min}(\mathbf{P}) - (1 - \cos(\varphi(t_0)))\sigma_{\max}(\mathbf{P}))}, \quad (9a)$$

$$(1 - \cos(\varphi(t_0))) < 2\frac{\sigma_{\min}(\mathbf{P})}{\sigma_{\max}(\mathbf{P})}, \quad (9b)$$

the attitude error is bounded by $\varphi(t) \leq \varphi_{\max} < \pi$ for all $t \geq t_0$.

Proof: Let $\mathbf{x} := (\tilde{\mathcal{R}}, \tilde{\mathbf{b}}_\omega)$, define the set $\Omega_\rho = \{\mathbf{x} \in \mathrm{SO}(3) \times \mathbb{R}^3 : V \leq \rho\}$, and the weighted distance of the state to the origin $d_0(\mathbf{x}) = \frac{1}{4}\|\mathbf{I} - \tilde{\mathcal{R}}\|^2\sigma_{\min}(\mathbf{P}) + \frac{1}{2k_{b_\omega}}\|\tilde{\mathbf{b}}_\omega\|^2$. The Lyapunov function is lower bounded by $V \geq d_0(\mathbf{x})$, so the set Ω_ρ is contained in the compact set defined by $d_0(\mathbf{x}) \leq \rho$ and thus is compact. The Lyapunov function verifies $\dot{V} \leq 0$ in Ω_ρ, so Ω_ρ is positively invariant. Consequently, $\forall_{t \geq t_0} d_0(\mathbf{x}(t)) \leq V(\mathbf{x}(t)) \leq V(\mathbf{x}(t_0))$ and the state is bounded for all $t \geq t_0$.

The conditions (9) imply that there exists $c_{\max}$ such that $V(\mathbf{x}(t_0)) \leq c_{\max} < 2\sigma_{\min}(\mathbf{P})$. Define $\varphi_{\max}$ such that $c_{\max} = \sigma_{\min}(\mathbf{P})(1 - \cos(\varphi_{\max}))$. The invariance of Ω_ρ yields $V(\mathbf{x}(t)) \leq V(\mathbf{x}(t_0))$ and, using $\frac{1}{4}\|\mathbf{I} - \tilde{\mathcal{R}}\|^2 = (1 - \cos(\varphi))$, produces $(1 - \cos(\varphi(t)))\boldsymbol{\phi}'\mathbf{P}\boldsymbol{\phi} \leq \sigma_{\min}(\mathbf{P})(1 - \cos(\varphi_{\max}))$, implying $(1 - \cos(\varphi(t))) \leq (1 - \cos(\varphi_{\max}))$ for all $t > t_0$, which shows that $\varphi(t) \leq \varphi_{\max}$ for all $t \geq t_0$. ■

By Lemma 5, the equilibrium point $(\tilde{\mathcal{R}}, \tilde{\mathbf{b}}_\omega) = (\mathbf{I}, \mathbf{0})$ is stable. Exponential stability is obtained in the following theorem.

Theorem 6: For any initial condition given by (9b), let the feedback gain satisfy (9a). Then the attitude and bias estimation errors converge exponentially fast to the stable equilibrium point $(\tilde{\mathcal{R}}, \tilde{\mathbf{b}}_\omega) = (\mathbf{I}, \mathbf{0})$.

Proof: Let the attitude error vector be given by $\tilde{\mathbf{q}}_q = \sin(\frac{\varphi}{2})\boldsymbol{\phi}$, the closed loop attitude and bias compensation errors kinematics are described by

$$\dot{\tilde{\mathbf{q}}}_q = -k_\omega \mathbf{Q}(\tilde{\mathbf{q}})\mathbf{Q}'(\tilde{\mathbf{q}})\mathbf{P}\tilde{\mathbf{q}}_q + \frac{1}{2}\mathbf{Q}(\tilde{\mathbf{q}})\tilde{\mathbf{b}}_\omega, \quad (10a)$$

$$\dot{\tilde{\mathbf{b}}}_\omega = -2k_{b_\omega}\mathbf{Q}'(\tilde{\mathbf{q}})\mathbf{P}\tilde{\mathbf{q}}_q, \quad (10b)$$

where $\mathbf{Q}(\tilde{\mathbf{q}}) = \tilde{q}_s\mathbf{I} + (\tilde{\mathbf{q}}_q)_\times$, $\tilde{q}_s = \cos(\frac{\varphi}{2})$, $\dot{\tilde{q}}_s = k_\omega \tilde{q}_s \tilde{\mathbf{q}}_q'\mathbf{P}\tilde{\mathbf{q}}_q - \frac{1}{2}\mathbf{q}_q'\tilde{\mathbf{b}}_\omega$, and $\tilde{\mathbf{q}} = \begin{bmatrix}\tilde{\mathbf{q}}_q' & \tilde{q}_s\end{bmatrix}'$ is the Euler quaternion representation [13]. Using $\|\tilde{\mathbf{q}}_q\|^2 = \frac{1}{8}\|\tilde{\mathcal{R}} - \mathbf{I}\|^2$, the Lyapunov function (5) in quaternion coordinates is described by $V = 2\tilde{\mathbf{q}}_q'\mathbf{P}\tilde{\mathbf{q}}_q + \frac{1}{2k_{b_\omega}}\|\tilde{\mathbf{b}}_\omega\|^2$.

Define the system (10) in the domain $\mathcal{D}_q = \{(\tilde{\mathbf{q}}_q, \tilde{\mathbf{b}}_\omega) \in \mathrm{B}(3) \times \mathbb{R}^3 : V \leq 2\sigma_{min}(\mathbf{P})(1 - \varepsilon_q)\}$, $0 < \varepsilon_q < 1$. The set $\mathcal{D}_q$ is given by the interior of the Lyapunov surface, so it is positively invariant and well defined. The feedback gain (9a) implies that any initial condition satisfying (9b) is in the set $\mathcal{D}_q$ for ε_q small enough.

Define the parameterized linear time-varying system

$$\begin{bmatrix}\dot{\tilde{\mathbf{q}}}_{q\star} \\ \dot{\tilde{\mathbf{b}}}_{\omega\star}\end{bmatrix} = \begin{bmatrix}\mathcal{A}(t,\lambda) & \mathcal{B}'(t,\lambda) \\ -\mathcal{C}(t,\lambda) & \mathbf{0}_{3\times3}\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{q}}_{q\star} \\ \tilde{\mathbf{b}}_{\omega\star}\end{bmatrix}, \quad (11)$$

where $(\tilde{\mathbf{q}}_{q\star}, \tilde{\mathbf{b}}_{\omega\star}) \in \mathbb{R}^3 \times \mathbb{R}^3$, $\lambda \in \mathbb{R}_{\geq 0} \times \mathcal{D}_q$. The matrices $\mathcal{A}(t,\lambda) := k_\omega \mathbf{Q}(\tilde{\mathbf{q}}(t,\lambda))\mathbf{Q}'(\tilde{\mathbf{q}}(t,\lambda))\mathbf{P}$, $\mathcal{B}(t,\lambda) := \frac{1}{2}\mathbf{Q}'(\tilde{\mathbf{q}}(t,\lambda))$ and $\mathcal{C}(t,\lambda) := 2k_{b_\omega}\mathbf{Q}'(\tilde{\mathbf{q}}(t,\lambda))\mathbf{P}$ are bounded, so the system is well defined. The quaternion $\tilde{\mathbf{q}}(t,\lambda)$ represents the solution of (10) with initial condition $\lambda = (t_0, \tilde{\mathbf{q}}_q(t_0), \tilde{\mathbf{b}}_\omega(t_0))$. If the parameterized LTV system (11) is λ-UGES, then the nonlinear system (10) is uniformly exponentially stable in the domain $\mathcal{D}_q$, see [15] for discussion and details. The parameterized LTV system verifies the assumptions of [15, Theorem 1]:

i) The elements of $\mathcal{B}(t,\lambda)$ and $\frac{\partial \mathcal{B}(t,\lambda)}{\partial t} = \frac{1}{2}\mathbf{Q}'(\dot{\tilde{\mathbf{q}}}(t,\lambda))$ are upper bounded, for all $\lambda \in \mathbb{R}_{\geq 0} \times \mathcal{D}_q$, $t \geq t_0$.

ii) The positive definite matrices $P(t,\lambda) = 4k_{b_\omega}\mathbf{P}$ and $Q(t,\lambda) = 8k_{b_\omega}k_\omega \mathbf{P}\mathbf{Q}(\tilde{\mathbf{q}}(t,\lambda))\mathbf{Q}'(\tilde{\mathbf{q}}(t,\lambda))\mathbf{P}$ satisfy $P(t,\lambda)\mathcal{B}'(t,\lambda) = \mathcal{C}'(t,\lambda)$, $p_m\mathbf{I} \leq P(t,\lambda) \leq p_M\mathbf{I}$, $-Q(t,\lambda) = \mathcal{A}'(t,\lambda)P(t,\lambda) + P(t,\lambda)\mathcal{A}(t,\lambda) + \dot{P}(t,\lambda)$, $q_m\mathbf{I} \leq Q(t,\lambda) \leq q_M\mathbf{I}$, with $p_m = 4k_{b_\omega}\sigma_{\min}(\mathbf{P})$, $p_M = 4k_{b_\omega}\sigma_{\max}(\mathbf{P})$, $q_m = 8k_\omega k_{b_\omega}\cos^2(\frac{\varphi_{\max}}{2})\sigma^2_{\min}(\mathbf{P})$ and $q_M = 8k_\omega k_{b_\omega}\sigma^2_{\max}(\mathbf{P})$.

The system (11) is λ-UGES if and only if $\mathcal{B}(t,\lambda)$ is λ-uniformly persistently exciting (λ-uPE) [15]. For any unitary norm vector $\mathbf{y}$, $4\mathbf{y}'\mathcal{B}(\tau,\lambda)\mathcal{B}'(\tau,\lambda)\mathbf{y} = \mathbf{y}'(\mathbf{I} - \tilde{\mathbf{q}}_q\tilde{\mathbf{q}}_q')\mathbf{y} \geq 1 - \|\tilde{\mathbf{q}}_q\|^2 = \|\tilde{\mathbf{q}}_s\|^2 \geq \cos^2\left(\frac{\varphi_{\max}}{2}\right)$ which satisfies the persistency of excitation condition. Consequently, the parameterized LTV (11) is λ-UGES, and the nonlinear system (10) is exponentially stable in the domain $\mathcal{D}_q$. Using $\|\tilde{\mathbf{q}}_q\|^2 = \frac{1}{8}\|\tilde{\mathcal{R}} - \mathbf{I}\|^2$ yields exponential stability of the nonlinear system (8). ∎

B. Almost Global Asymptotic Stability

The trajectories of the attitude observer converge exponentially fast for any initial condition in a region characterized by (9). In this section, the convergence of the trajectories of the system emanating from anywhere in the domain is studied.

By Lemma 3, the equilibrium points of the system (8) are contained in $C_{\mathcal{R}}$. By substitution in (8), the largest invariant set in $C_{\mathcal{R}}$ is given by

$$I_{\mathcal{R}} = \{\tilde{\mathcal{R}}, \tilde{\mathbf{b}}_\omega \in \mathrm{SO}(3)\times\mathbb{R}^3 : (\tilde{\mathcal{R}}, \tilde{\mathbf{b}}_\omega) = (\mathbf{I}, \mathbf{0}) \vee (\tilde{\mathcal{R}}, \tilde{\mathbf{b}}_\omega) = (\mathrm{rot}(\pi, \mathrm{eigvec}(\mathbf{P})), \mathbf{0})\}.$$

The next theorem shows that, among all the equilibrium points in $I_{\mathcal{R}}$, only the origin is stable, which guarantees aGAS of the origin and, using Theorem 6, exponential convergence within a explicit region in the state space.

Theorem 7: Define $\mathbf{W}$ such that the eigenvalues of $\mathbf{P}$ are distinct. Under Assumption 1, the equilibrium point $(\tilde{\mathcal{R}}, \tilde{\mathbf{b}}_\omega) = (\mathbf{I}, \mathbf{0})$ of the system (8) is aGAS. Furthermore, every system solution emanating from the region of attraction of $(\tilde{\mathcal{R}}, \tilde{\mathbf{b}}_\omega) = (\mathbf{I}, \mathbf{0})$ converges exponentially fast for $t \geq t_e \geq t_0$, where t_e is the time instant that verifies $V(\tilde{\mathcal{R}}(t_e), \tilde{\mathbf{b}}(t_e)) \leq 2\sigma_{\min}(\mathbf{P})$.

Proof: By LaSalle's invariance principle, the trajectories of the system converge to the set $I_{\mathcal{R}}$. The equilibrium points contained in $I_{\mathcal{R}}$ are characterized using a local analysis, based on the local parametrization adopted in [6, Section 5], given by the first order terms of the DCM formulation

$$\tilde{\mathcal{R}} \approx \tilde{\mathcal{R}}^*(\mathbf{I} + (\eta)_\times), \quad \tilde{\mathbf{b}}_\omega \approx \tilde{\mathbf{b}}^*_\omega + \delta\mathbf{b}, \qquad (12)$$

where $\eta, \delta\mathbf{b} \in \mathbb{R}^3$, $\tilde{\mathcal{R}}^* = \mathrm{rot}(\pi, \boldsymbol{\phi}^*_i)$, $\boldsymbol{\phi}^*_i \in \mathrm{eigvec}(\mathbf{P})$, $\tilde{\mathbf{b}}^*_\omega = \mathbf{0}$ and $i = 1,2,3$ is the index of the equilibrium point. Applying (12) in the system (8) and neglecting second order terms produces $\begin{bmatrix}\dot{\eta}\\ \dot{\delta\mathbf{b}}\end{bmatrix} = \begin{bmatrix}k_\omega(\tilde{\mathcal{R}}^*\mathbf{W} - \mathrm{tr}(\tilde{\mathcal{R}}^*\mathbf{W})\mathbf{I}) & \mathbf{I}\\ k_{b_\omega}(\tilde{\mathcal{R}}^*\mathbf{W} - \mathrm{tr}(\tilde{\mathcal{R}}^*\mathbf{W})\mathbf{I}) & \mathbf{0}\end{bmatrix}\begin{bmatrix}\eta\\ \delta\mathbf{b}\end{bmatrix}$. Let the eigenvalues of $\mathbf{W}$ and $\mathbf{P}$ be denoted by $\alpha_{W\,l}$ and $\alpha_{P\,l}$, $l = 1,2,3$, respectively, with $\alpha_{W\,1} > \alpha_{W\,2} > \alpha_{W\,3}$ and $\alpha_{P\,1} > \alpha_{P\,2} > \alpha_{P\,3}$. From the definition of $\mathbf{P}$, the eigenvectors of $\mathbf{P}$ and $\mathbf{W}$ are equal and the eigenvalue i satisfies $\alpha_{P\,i} = \alpha_{W\,k} + \alpha_{W\,j}$, where i, k and j are distinct. Using $\tilde{\mathcal{R}}^* = -\mathbf{I} + 2\boldsymbol{\phi}^*_i\boldsymbol{\phi}^{*\prime}_i$, the spectral decomposition $\mathbf{W} = \sum_{l=1}^3 \alpha_{W\,l}\boldsymbol{\phi}^*_l\boldsymbol{\phi}^{*\prime}_l$, and defining $\mathbf{U} = \begin{bmatrix}\boldsymbol{\phi}^*_i & \boldsymbol{\phi}^*_j & \boldsymbol{\phi}^*_k\end{bmatrix} \in \mathrm{O}(3)$, produces $\tilde{\mathcal{R}}^*\mathbf{W} - \mathrm{tr}(\tilde{\mathcal{R}}^*\mathbf{W})\mathbf{I} = 2\boldsymbol{\phi}^*_i\boldsymbol{\phi}^{*\prime}_i\alpha_{W\,i} - \mathbf{W} - (\alpha_{Wj} + \alpha_{Wk} - \alpha_{W\,i}) = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}'$ where $\boldsymbol{\Lambda} = \mathrm{diag}(\alpha_{W\,j} + \alpha_{W\,k}, \alpha_{W\,k} - \alpha_{W\,i}, \alpha_{W\,j} - \alpha_{W\,i})$. The linearized system can be rewritten as $\begin{bmatrix}\dot{\eta}\\ \dot{\delta\mathbf{b}}\end{bmatrix} = \begin{bmatrix}k_\omega\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}' & \mathbf{I}\\ k_{b_\omega}\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}' & \mathbf{0}\end{bmatrix}\begin{bmatrix}\eta\\ \delta\mathbf{b}\end{bmatrix}$. The eigenvalues of $\left[\begin{smallmatrix}k_\omega\boldsymbol{\Lambda} & \mathbf{I}\\ k_{b_\omega}\boldsymbol{\Lambda} & \mathbf{0}\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}k_\omega\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}' & \mathbf{I}\\ k_{b_\omega}\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}' & \mathbf{0}\end{smallmatrix}\right]$ are equal and given by $\alpha_l = \frac{1}{2}(k_\omega[\boldsymbol{\Lambda}]_{ll} + \sqrt{k^2_\omega[\boldsymbol{\Lambda}]^2_{ll} + 4k_{b_\omega}[\boldsymbol{\Lambda}]_{ll}})$, $\alpha_{l+3} = \frac{1}{2}(k_\omega[\boldsymbol{\Lambda}]_{ll} - \sqrt{k^2_\omega[\boldsymbol{\Lambda}]^2_{ll} + 4k_{b_\omega}[\boldsymbol{\Lambda}]_{ll}})$, where $l = 1,2,3$, and $[\boldsymbol{\Lambda}]_{ll}$ denotes l^{th} diagonal element of $\boldsymbol{\Lambda}$. The real part of α_1 is always positive, the real parts of α_4, α_5 and α_6 are always negative, and the real parts of α_2 and α_3 are always nonzero. Therefore, the equilibrium points are hyperbolic and unstable. By the Stable-Unstable Manifold theorem and the Hartman-Grobman theorem [6], [16], the stable manifold of $(\tilde{\mathcal{R}}^*, \tilde{\mathbf{b}}^*_\omega) = (\mathrm{rot}(\pi, \mathrm{eigvec}(\mathbf{P})), \mathbf{0})$ has zero measure in $\mathrm{SO}(3)\times\mathbb{R}^3$ and the complement of the stable manifold is open and dense. Consequently, almost all initial conditions in $\mathrm{SO}(3)\times\mathbb{R}^3$ converge to the stable equilibrium point $(\tilde{\mathcal{R}}, \tilde{\mathbf{b}}_\omega) = (\mathbf{I}, \mathbf{0})$. Exponential convergence is obtained by using aGAS to show that the solutions of (8) enter the positively invariant set $V \leq 2\sigma_{\min}(\mathbf{P})$ for some $t_e \geq t_0$, where exponential stability is guaranteed by Theorem 6. ∎

Interestingly enough, in most practical application it is possible to guarantee upper bounds for the initial estimation errors. In that case, exponential stability of the origin for all valid initial conditions follows directly from Theorem 6.

Corollary 8: Assume that the initial estimation errors are bounded according to

$$\varphi(t_0) \leq \varphi_{0\max}, \quad \|\tilde{\mathbf{b}}_\omega(t_0)\| \leq b_{0\max}, \qquad (13)$$

where $(1 - \cos(\varphi_{0\max})) < 2\frac{\sigma_{\max}(\mathbf{P})}{\sigma_{\min}(\mathbf{P})}$, and select k_{b_ω} such that $k_{b_\omega} > b^2_{0\max}(4\sigma_{\min}(\mathbf{P}) - 2(1 - \cos(\varphi_{0\max}))\sigma_{\max}(\mathbf{P}))^{-1}$. Then the origin $(\tilde{\mathcal{R}}, \tilde{\mathbf{b}}_\omega) = (\mathbf{I}, \mathbf{0})$ is uniformly exponentially stable in the set defined by (13).

C. Output Feedback Configuration

This section describes how the attitude observer can be implemented using directly the sensor measurements and state estimates.

Proposition 9: The feedback laws (6) and (7) are explicit functions of the sensor readings and state estimates

$$\hat{\boldsymbol{\omega}} = f_{\mathcal{R}}(\mathbf{H}_r, \hat{\mathcal{R}})(\boldsymbol{\omega}_r - \hat{\mathbf{b}}_\omega) + k_\omega \mathbf{s}_{\mathcal{R}}, \quad \dot{\hat{\mathbf{b}}}_\omega = k_{b_\omega}\mathbf{s}_{\mathcal{R}},$$
$$\mathbf{s}_{\mathcal{R}} = (f_{\mathcal{R}}(\mathbf{H}_r, \hat{\mathcal{R}})\mathbf{W} - \mathbf{W}f'_{\mathcal{R}}(\mathbf{H}_r, \hat{\mathcal{R}}))_\otimes,$$

where $f_{\mathcal{R}}(\mathbf{H}_r, \hat{\mathcal{R}}) := \hat{\mathcal{R}}'\mathbf{H}\mathbf{A}_H\mathbf{A}'_H\mathbf{H}'_r$ and $\mathbf{H}_r := \begin{bmatrix}\mathbf{h}_{r\,1} & \cdots & \mathbf{h}_{r\,n}\end{bmatrix}$.

Proof: Using ${}^B\hat{\mathbf{U}}_H{}^B\mathbf{U}'_H = \hat{\mathcal{R}}'\mathbf{U}_H\mathbf{U}'_H\mathcal{R} = \tilde{\mathcal{R}}$ yields $\mathbf{s}_{\mathcal{R}} = ({}^B\hat{\mathbf{U}}_H{}^B\mathbf{U}'_H\mathbf{W} - \mathbf{W}{}^B\mathbf{U}_H{}^B\hat{\mathbf{U}}'_H)_\otimes$. Applying $\hat{\mathbf{U}}_H := \hat{\mathcal{R}}'\mathbf{H}\mathbf{A}_H$, $\mathbf{H}_r = \mathcal{R}'\mathbf{H}$ and $\mathbf{H}_r\mathbf{A}_H = {}^B\mathbf{U}_H$ produces the desired results. ■

V. OBSERVER STABILITY WITH RATE GYRO NOISE

In this section, the stability of the attitude observer in the presence of bounded time-varying disturbances in the rate gyro measurements is studied. Although the origin of the unforced system is aGAS, generic exogenous disturbances may drive the trajectories of system to the unstable equilibrium points. Sufficient conditions for ultimate boundedness of the attitude estimate in the presence of noise in the inertial sensor are provided.

The rate gyro sensor measurements are described by

$$\boldsymbol{\omega}_r = \boldsymbol{\omega} + \mathbf{n}_\omega,$$

where $\mathbf{n}_\omega$ is the sensor noise, bounded by $n_{max} \geq \|\mathbf{n}_\omega\|$. Using the feedback law $\hat{\boldsymbol{\omega}} = \tilde{\mathcal{R}}\boldsymbol{\omega}_r + k_\omega\mathbf{s}_{\mathcal{R}}$, the closed loop kinematics of the attitude error are given by

$$\dot{\tilde{\mathcal{R}}} = k_\omega\tilde{\mathcal{R}}(\tilde{\mathcal{R}}'\mathbf{W} - \mathbf{W}\tilde{\mathcal{R}}) - \tilde{\mathcal{R}}(\mathbf{n}_\omega)_\times, \tag{14}$$

which are well defined on $\mathrm{SO}(3)$ in spite of the presence of the rate gyro sensor disturbance. The stability of the observer in the presence of inertial sensor noise is presented next.

Theorem 10: Choose $\varphi_{\min} \in \begin{pmatrix}0 & \frac{\pi}{2}\end{pmatrix}$, let k_ω satisfy

$$k_\omega > \frac{n_{\max}}{\sin(\varphi_{\min})\sigma_{\min}(\mathbf{P})}, \tag{15}$$

and let $\mathbf{W}$ be such that $(1 - \cos(\varphi_{\max})) \geq \frac{\sigma_{\max}(\mathbf{P})}{\sigma_{\min}(\mathbf{P})}(1 - \cos(\varphi_{\min}))$ is verified, where $\varphi_{\max} = \pi - \varphi_{\min}$. Then for any initial condition such that

$$\|\mathbf{I} - \tilde{\mathcal{R}}(t_0)\|^2 < 4\frac{\sigma_{\min}(\mathbf{P})}{\sigma_{\max}(\mathbf{P})}(1 - \cos(\varphi_{\max})), \tag{16}$$

there exists T such that the trajectory of the system (14) satisfies

$$\|\mathbf{I} - \tilde{\mathcal{R}}(t)\|^2 \leq 4\frac{\sigma_{\max}(\mathbf{P})}{\sigma_{\min}(\mathbf{P})}(1 - \cos(\varphi_{\min})), \tag{17}$$

for all $t \geq t_0 + T$.

Proof: The proof is based on the derivation of boundedness properties for nonlinear systems presented in [17, Theorem 4.18]. Using the Lyapunov function V_H defined in (4) yields $\dot{V}_H = -k_\omega\|s_{\mathcal{R}}\|^2 - s'_{\mathcal{R}}\tilde{\mathcal{R}}\mathbf{n}_\omega \leq -\|s_{\mathcal{R}}\|(k_\omega\|s_{\mathcal{R}}\| - n_{\max})$. Using the algebraic manipulations adopted in [12] produces $\|s_{\mathcal{R}}\| = \|\mathbf{Q}'_s(\varphi, \boldsymbol{\phi})\mathbf{P}\boldsymbol{\phi}\| \geq \sigma_{\min}(\mathbf{Q}'_s(\varphi, \boldsymbol{\phi}))\sigma_{\min}(\mathbf{P}) \geq \sin(\varphi)\sigma_{\min}(\mathbf{P})$, where $\mathbf{Q}_s(\varphi, \boldsymbol{\phi}) = \sin(\varphi)\mathbf{I} + (1 - \cos(\varphi))(\boldsymbol{\phi})_\times$. The gain condition (15) produces

$$\begin{aligned}\dot{V}_H &< -\|s_{\mathcal{R}}\|n_{\max}\left(\frac{\|s_{\mathcal{R}}\|}{\sin(\varphi_{\min})\sigma_{\min}(\mathbf{P})} - 1\right)\\ &= -\|s_{\mathcal{R}}\|n_{\max}\left(\frac{\sin(\varphi)}{\sin(\varphi_{\min})} - 1\right),\end{aligned}$$

and $\varphi \in \begin{bmatrix}\varphi_{\min} & \varphi_{\max}\end{bmatrix} \Rightarrow \dot{V}_H < 0$. Let $\mu := \|\mathbf{I} - \mathcal{R}_{\min}\|^2$, $\mathcal{R}_{\min} := \mathrm{rot}(\varphi_{\min}, \boldsymbol{\phi})$ and $r := \|\mathbf{I} - \mathcal{R}_{\max}\|^2$, $\mathcal{R}_{\max} := \mathrm{rot}(\varphi_{\max}, \boldsymbol{\phi})$, where $\boldsymbol{\phi}$ is arbitrary, then $\|\mathbf{I} - \tilde{\mathcal{R}}\|^2 \in \begin{bmatrix}\mu & r\end{bmatrix} \Rightarrow \dot{V}_H < 0$, which characterizes a compact set in $\mathrm{SO}(3)$ where the Lyapunov function decreases along the system trajectories.

The Lyapunov function satisfies $\alpha_1(\|\mathbf{I} - \tilde{\mathcal{R}}\|^2) \leq V_H \leq \alpha_2(\|\mathbf{I} - \tilde{\mathcal{R}}\|^2)$ where $\alpha_1(x) = \frac{1}{4}\sigma_{\min}(\mathbf{P})x$, $\alpha_2(x) = \frac{1}{4}\sigma_{\max}(\mathbf{P})x$. Define the sets $\Omega_{t,\mu} = \{\tilde{\mathcal{R}} \in \mathrm{SO}(3) : V_H \leq \alpha_2(\mu)\}$, $\Omega_{t,r} = \{\tilde{\mathcal{R}} \in \mathrm{SO}(3) : V_H \leq \alpha_1(r)\}$, and $\mathrm{C}_{t,\mu,r} = \{\Omega_{t,r} - \Omega_{t,\mu}\}$ which is nonempty by construction of $\mathbf{W}$. Any point $\tilde{\mathcal{R}} \in \mathrm{C}_{t,\mu,r}$ satisfies $\|\mathbf{I} - \tilde{\mathcal{R}}\|^2 \in \begin{bmatrix}\mu & r\end{bmatrix}$, as shown by using

$$\begin{aligned}&V_H \leq \alpha_1(r) \Rightarrow \alpha_1(\|\mathbf{I} - \tilde{\mathcal{R}}\|^2) \leq \alpha_1(r) \Rightarrow \|\mathbf{I} - \tilde{\mathcal{R}}\|^2 \leq r,\\ &V_H \geq \alpha_2(\mu) \Rightarrow \alpha_2(\|\mathbf{I} - \tilde{\mathcal{R}}\|^2) \geq \alpha_1(r) \Rightarrow \|\mathbf{I} - \tilde{\mathcal{R}}\|^2 \geq \mu,\end{aligned}$$

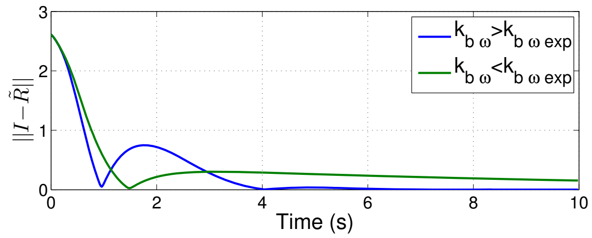
and hence $\tilde{\mathcal{R}} \in \mathrm{C}_{t,\mu,r} \Rightarrow \dot{V}_H < 0$. Because $\dot{V}_H$ is strictly negative in $\mathrm{C}_{t,\mu,r}$, any solution starting in $\mathrm{C}_{t,\mu,r}$ will reach $\Omega_{t,\mu}$ in finite time and any solution starting in $\Omega_{t,\mu}$ will remain in the set since $\dot{V}_H < 0$ in the corresponding boundary, see [17, Section 4.8] for a motivation of the level sets involved. The initial conditions given by (16) satisfy $\tilde{\mathcal{R}}(t_0) \in \Omega_{t,r}$; any $\tilde{\mathcal{R}} \in \Omega_{t,\mu}$ satisfies (17), which concludes the proof. ■

The conditions of Theorem 10 are of interest in practical applications, since they allow for worst-case attitude errors to be driven to a desired neighborhood of the origin, by appropriate choice of k_ω and $\mathbf{W}$.

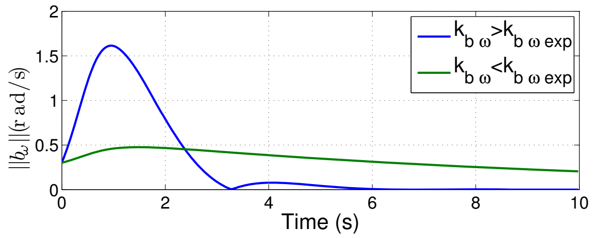
VI. SIMULATIONS

In this section, simulation results for the proposed attitude observer are presented. The directions of the vector measurements are given by ${}^L\mathbf{h}_1 = \begin{bmatrix}1 & 0 & 0\end{bmatrix}'$ and ${}^L\mathbf{h}_2 = \begin{bmatrix}0 & 0 & 1\end{bmatrix}'$, which are a simplified representation of vectors sensed in frame $\{B\}$ by a magnetic compass and a pendulum, respectively, and satisfy the conditions expressed in Proposition 2.

The attitude observer stability in the presence of rate gyro bias is studied first. The observer parameters are given by $k_\omega = 1$, $k_{b_\omega} \in \{10^{-1}, 1\}$, $\mathbf{W} = \begin{bmatrix}1.1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0.9\end{bmatrix}$, $\varphi(t_0) = \frac{3\pi}{4}$ rad, $\mathbf{b}_\omega = \frac{10\pi}{180}\mathbf{1}$ rad/s, $\hat{\mathbf{b}}_\omega(t_0) = \mathbf{0}$ rad/s, where the initial conditions are realistic for most practical applications. The rigid body trajectory is computed using oscillatory angular rates of 1 Hz. The attitude and bias estimation results are depicted in Fig. 1, where $k_{b_\omega\,\exp} = 0.21$ is the minimum feedback gain (9a) that guarantees exponential convergence. The trajectories convergences faster for higher gain k_{b_ω}, as expected. The peak of the bias estimate for large k_{b_ω} is

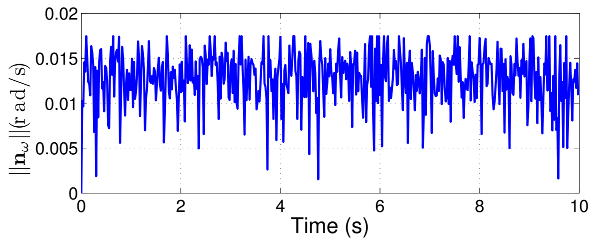


(a) Attitude estimation error.

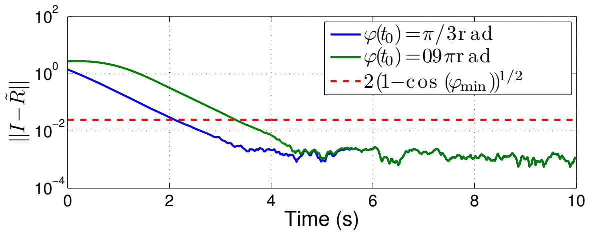


(b) Bias estimation error.

Fig. 1. Observer stability with bias in the rate gyro measurements.



(a) Rate gyro noise.



(b) Attitude estimation error.

Fig. 2. Observer stability with noise in the rate gyro measurements.

justified by the level set $V \leq c$ of the Lyapunov function (5) with $\gamma_b = k_{b_\omega}^{-1}$ containing the points satisfying $\|\tilde{\mathbf{b}}_\omega\|^2 \approx 2k_{b_\omega} c$, $\|\mathbf{I} - \tilde{\mathcal{R}}\|^2 \approx 0$.

The stability properties of the attitude observer in the presence of rate gyro noise are illustrated by considering low cost device readings, corrupted by the disturbance depicted in Fig. 2(a) and bounded by $\|\mathbf{n}_\omega\| < n_{\max} = 1.75 \times 10^{-2}$ rad/s. The observer parameters are designed to guarantee $\varphi_{\min} = \frac{\pi}{180}$ rad and are given by $k_\omega = k_{\omega\,\min} + 10^{-4}$, $\mathbf{W} = \mathbf{I}$, $\varphi_{\max} = \frac{179\pi}{180}$ rad, where $k_{\omega\,\min} = 1 + 10^{-4}$ is the minimum gain that verifies (15). The simulation results for $\varphi(t_0) = \frac{\pi}{3}$ rad and $\varphi(t_0) = \frac{9\pi}{10}$ rad are depicted in Figure 2(b) using a logarithmic scale in the attitude error axis. For both initial conditions, the attitude error converges in finite time to the region given by (17), as desired.

VII. CONCLUSIONS

A nonlinear observer for attitude estimation on $SO(3)$ was derived. Almost global asymptotic stability and exponential convergence of the attitude estimates in the presence of biased rate gyro readings were demonstrated. Boundedness of the attitude estimation error in the presence of rate gyro noise was shown. The stability and convergence results were formulated in terms of the design parameters, which can be determined to satisfy accuracy specifications in practical applications. Simulations results illustrated the stability of the observer. Future work will address the stability analysis for the case where both angular rate noise and bias are present, and exploiting the observer design parameters to tackle sensor noise in practical setups.

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