Multiple Model Adaptive Estimation and Model Identification using a Minimum Energy Criterion

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***Abstract*— This paper addresses the problem of Multiple Model Adaptive Estimation (MMAE) for discrete-time, linear, time-invariant MIMO plants with parameter uncertainty and unmodeled dynamics. Model identification is analyzed in a deterministic setting by adopting a Minimum Energy selection criterion. The MMAE system relies on a finite number of local observers, each designed using a selected model (SM) from the original set of possibly infinite plant models. Results akin to those previously obtained in a stochastic setting are derived in a far simpler manner, in a deterministic framework. We show, under suitable distinguishability conditions, that the SM identified is the one that corresponds to the observer with smallest output prediction error energy. We also develop a procedure to analyze the behavior of MMAE when the true plant is not one of the SMs. This leads to an algorithm that computes, for each SM, the set of equivalently identified plants, that is, the set of plants that will be identified as that particular SM. The impact of unmodeled dynamics on model identification is discussed. Simulation results with a model of a motor coupled to a load via an elastic shaft illustrate the performance of the methodology derived.**

I. INTRODUCTION

The design of a single state-observer for a given plant requires exact knowledge of the plant parameters for superior performance. In practice, parameter uncertainty will impact the performance and robustness of the observer. In fact, incorrect modeling in the observer design may lead to large estimation errors or even error divergence [1]. To cope with this problem, adaptive estimation algorithms (where the adaptation is with respect to the uncertainty in the plant parameters) have been proposed in the literature. Among these, the Multiple Model Adaptive Estimation (MMAE) algorithm has received special attention [2]–[5]. However, the use of multiple models for Adaptive Estimation goes back to the 1960s and 1970s when several authors including [2], [3], studied Kalman filter based estimators.

In the stochastic version of the MMAE [2]–[4], a separate discrete-time Kalman filter (KF) is developed for each "selected model" (SM) defined by an hypothesized parameter vector in the unknown parameter set. The resulting set of KFs forms a "bank" where each local KF generates its own state estimate and an output error (residual), as shown in Fig. 1. The bank of KFs runs in parallel and at each sampling instant the MMAE uses a nonlinear function of the measurement residuals of each SM to compute the conditional probability p^i that the filter selected be the one corresponding to the true plant model. The state estimate is a probabilistically weighted combination of each KF estimate. The rationale is that the highest probability should be assigned to the state estimation provided by the most accurate KF, and lower probabilities assigned to the remaining KFs.

In the last decade, MMAE has been the subject of considerable research effort that is patent in a vast number of publications; see [5]–[9] and the references therein. MMAE is at the root of many techniques for estimation, navigation, tracking, and surveillance. It is also the basis for Multiple-Model Adaptive Control, see [7], [10]–[16].

In [4], by introducing an information theoretic measure, the authors analyzed the convergence of the conditional probabilities p^i and showed that the one corresponding to the KF designed for the closest to the actual system (in a stochastic norm sense) converges to one, while the others tend to zero. The theoretical setup exploited in [4] requires extensive knowledge of stochastic analysis, information theory, and measure theory. Here, we show that similar results can be obtained by resorting to a much simpler deterministic framework that relies on the use of Krener Observers (KO) [17] rather than Kalman Filters. We prove that under suitable distinguishability conditions, the model identified is the one that corresponds to the observer exhibiting the smallest output prediction error energy.

Other contributions of this paper are the introduction of the concept of set of Equivalently Identified Plants (EIP), defined as the set of plants that will be identified as the same SM, and an algorithm to compute it. We also analyze the impact of unmodelled dynamics (unstructured uncertainty) on model identification and in particular how this translates into the topology of the EIP sets.

The structure of the paper is as follows. In section II we review the main issues of MMAE and define the structure of a Minimum Energy (ME) MMAE. Section III summarizes our main results. The effect of unmodeled dynamic upon model identification is considered in Section IV. Section V illustrates the performance of the ME-MMAE algorithm proposed, through computer simulations with a model of a motor coupled to a load via an elastic shaft. Conclusions and suggestions for future research are summarized in Section VI.

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II. THE MULTIPLE-MODEL ADAPTIVE ESTIMATOR

This section introduces a class of MMAEs in a deterministic setting. MMAE relies on a finite number N of selected models chosen from the original set of (possibly infinite) plant models and consists of: i) a dynamic generator of N weighting signals and ii) a bank of N discrete-time observers, where each observer is designed based on one of the SMs adopted. The state estimate is generated by a weighted sum of the local state-estimates produced by the bank of observers. The dynamic weights are provided by a difference dynamic equation called the Dynamic Weighting Signal Generator (DWSG). Fig. 1 shows the structure of the MMAE in which the plant is described by an LTI difference equation, ξ_t and θ_t are deterministic plant and measurement noise sequences respectively, and the observers are designed using different values of the uncertain parameters. We assume the plant model G is subjected to parameter uncertainty $\kappa \in \mathbb{R}^l$, that is, $G = G(\kappa)$. In what follows we consider multiple-input-multiple-output (MIMO) linear time-invariant (LTI) plant models of the form

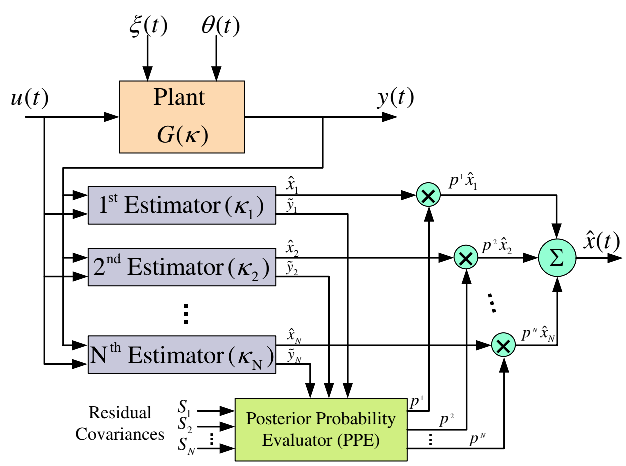

Fig. 1. The MMAE architecture

$$x_{t+1} = A_\kappa x_t + B_\kappa u_t + L_\kappa \xi_t, \tag{1a}$$
$$y_t = C_\kappa x_t + \theta_t, \tag{1b}$$

where $x_t \in \mathbb{R}^n$ denotes the state of the system, $u_t \in \mathbb{R}^m$ its control input, $y_t \in \mathbb{R}^q$ its measured noisy output, $\xi_t \in \mathbb{R}^r$ an input plant disturbance that can not be measured, and $\theta_t \in \mathbb{R}^q$ is the measurement noise. The initial condition x_0 of (1) and the signals ξ_t and θ_t are assumed unknown but bounded. The matrices A_κ, B_κ, L_κ, and C_κ contain *unknown constant parameters* denoted by vector κ.
Consider a finite set of candidate parameter values $\mathsf{K} := \{\kappa_1, \kappa_2, \ldots, \kappa_N\}$ indexed by $i \in \{1, \ldots, N\}$. We propose the following MMAE. The state estimate is given by

$$\hat{x}_t := \sum_{i=1}^{N} p_t^i \hat{x}_{t|\kappa_i}, \tag{2}$$
$$\hat{y}_t := \sum_{i=1}^{N} p_t^i \hat{y}_{t|\kappa_i}, \tag{3}$$
$$\hat{\kappa}_t := \kappa_{i^\star}, \quad i^\star := \arg \max_{i\in\{1,\ldots,N\}} p_t^i, \tag{4}$$

where $\hat{x}_t$, $\hat{y}_t$ and $\hat{\kappa}_t$ are the estimates of the state x, output y_t, and parameter vector κ at time t, respectively and p_t^i are dynamic weights (which are defined below). In (2), each $\hat{x}_{t|\kappa_i}$; $i = 1, \ldots, N$ corresponds to a "local" state estimate generated by the i^{th} (steady state) Krener minmax observer [17]

$$\hat{x}_{t+1|\kappa_i} = A_{\kappa_i}\hat{x}_{t|\kappa_i} + B_{\kappa_i}u_t + H_{\kappa_i}\big(y_t - C_{\kappa_i}\hat{x}_{t|\kappa_i}\big), \tag{5a}$$
$$\hat{y}_{t|\kappa_i} = C_{\kappa_i}\hat{x}_{t|\kappa_i}, \tag{5b}$$
$$H_{\kappa_i} = \Sigma_{\kappa_i}C_{\kappa_i}^T[C_{\kappa_i}\Sigma_{\kappa_i}C_{\kappa_i}^T + \Theta]^{-1} \tag{5c}$$

where Σ_{κ_i} is the solution of the discrete Riccati equation

$$\begin{aligned} 0 = &-\Sigma_{\kappa_i} + A_{\kappa_i}\Sigma_{\kappa_i}A_{\kappa_i}^T + L_{\kappa_i}\Xi L_{\kappa_i}^T \\ &- A_{\kappa_i}^T\Sigma_{\kappa_i}C_{\kappa_i}^T[C_{\kappa_i}\Sigma_{\kappa_i}C_{\kappa_i}^T + \Theta]^{-1}C_{\kappa_i}\Sigma_{\kappa_i}A_{\kappa_i}, \end{aligned} \tag{6}$$

and it is assumed that $[A_{\kappa_i}, L_{\kappa_i}]$ and $[A_{\kappa_i}, C_{\kappa_i}]$ for $i = 1, \ldots, N$ are controllable and observable, respectively. The symmetric positive definite weighting matrices Ξ and Θ are viewed as parameters to be chosen based on information about the disturbance, the measurement noise, and the plant.

In the sequel we introduce dynamic weights which weigh the local estimations (2) and dictate the estimation of the uncertain parameter (4).

A. Dynamic Weighting Signal Generator (DWSG)

To generate the dynamic weights p^i , we use the dynamic recursion

$$p_{t+1}^i = \frac{\beta_i e^{-w_t^i}}{\sum_{j=1}^N p_t^j \beta_j e^{-w_t^j}} p_t^i, \tag{7}$$

where β_i is a positive weighting constant matrix and w_t^i is a continuous function called an *error measuring function* that maps the measurable signals of the plant and the states of the i^{th} local observer to a nonnegative real value. Examples of an error measuring function and a β_i that we use in this paper are

$$w_t^i := \frac{1}{2}\|y_t - \hat{y}_{t|\kappa_i}\|^2_{S_{\kappa_i}^{-1}}, \qquad \beta_i := \frac{1}{\sqrt{|S_{\kappa_i}|}}, \tag{8}$$

where S_{κ_i} is a positive definite weighting matrix and $\|x\|_S = \left(x^T S x\right)^{\frac{1}{2}}$. The matrices S_{κ_i} are important to scale the energy of the estimation error sequences in order to make them comparable. They are computed as

$$S_{\kappa_i} = C_{\kappa_i}\Sigma_{\kappa_i}^{-}C_{\kappa_i}^T + \Theta. \tag{9}$$

For stable dynamic plants, S_{κ_i} is a positive definite matrix. The structure of the key equation (7), which generates the time-sequence of the weights p^i, is inspired by the "standard" stochastic discrete-time MMAE formulas (see [3]). We impose the constraint that the initial conditions p_0^i be chosen such that $p_0^i \in (0,1)$ and $\sum_{i=1}^N p_0^i = 1$.

Remark 1. Throughout this paper we formulate the ME-MMAE in a deterministic setting; however, the same method is applicable to stochastic systems. In the stochastic setup [2]–[4], the signals ξ_t and θ_t are assumed zero mean independent discrete-time white noise sequence with covariances $cov[\xi_t;\xi_\tau] = \Xi\delta_{t\tau}$ and $cov[\theta_t;\theta_\tau] = \Theta\delta_{t\tau}$, respectively and (1) is initialized at $t = 0$ with $E\{x_0\} = 0$ and $E\{x_0 x_0^T\} = \Sigma_0$ and the S_{κ_i} in (9) represents the covariance matrix of residuals and the observers are Kalman filters (in fact, in [17] it is shown that the Kalman filter is also a minimax filter). The advantage of using the deterministic MMAE version is that it is easier to investigate stability and performance-robustness issues as compared to the stochastic version of the problem. □

III. MAIN RESULTS

In this section we summarize our main results on identification and convergence of the ME-MMAE. We first show that positiveness and boundedness of the dynamic weights p_t^i are independent of the input signals of the DWSG system. We also show that the overall sum of the p^i's is always unity for all $t \geq 0$.

Theorem 1. *Suppose that the initial conditions p_0^i satisfy $p_0^i \in (0,1)$ and $\sum_{i=1}^N p_0^i = 1$. Then, each p_t^i, $i = 1,\dots N$ governed by (7) is nonnegative, uniformly bounded and contained in the interval $[0,1]$ for every $t \geq 0$. Furthermore,*

$$\sum_{i=1}^N p_t^i = 1, \quad \forall t \geq 0$$

Proof. Defining $P_t := \sum_{i=1}^N p_t^i$ and computing its time-evolution using (7) yields

$$\begin{aligned} P_{t+1} &= \sum_{i=1}^N \frac{\beta_i e^{-w_t^i}}{\sum_{j=1}^N p_t^j \beta_j e^{-w_t^j}} p_t^i \\ &= \frac{\sum_{i=1}^N p_t^i \beta_i e^{-w_t^i}}{\sum_{j=1}^N p_t^j \beta_j e^{-w_t^j}} = 1 \end{aligned}$$

Therefore, if $P_0 = 1$, then $P_t = 1$, $\forall t \geq 0$. We now show, that if $p_0^i > 0$, then $p_t^i \geq 0$, $\forall t \geq 0$. From (7), if $p_0^i > 0$, then p_t^i cannot be negative. The boundedness condition $p_t^i \in [0,1]$, $\forall t \geq 0$ follows immediately from the fact that $p_t^i \geq 0$ and $P_t = 1$ for all $t \geq 0$. □

We next provide conditions for convergence of the dynamic weights p_t^i.

Theorem 2. *Let $i^\star \in \{1,2,\dots,N\}$ be an index of a parameter vector in κ and let $\mathcal{I} := \{1,2,\dots,N\}\backslash i^\star$ an index set. Suppose that there exist positive constants n_1, t_1, ϵ, and ϵ_1 such that for all $t \geq t_1$ and $n \geq n_1$ the following condition holds:*

$$\frac{1}{n}\sum_{\tau=t}^{t+n-1}\left((w_\tau^{i^\star}) + \epsilon\right) < \frac{1}{n}\sum_{\tau=t}^{t+n-1}\min_{j\in\mathcal{I}} w_\tau^j, \tag{10}$$

where w^i are defined in (8) and

$$\left(\ln\max_{j\in\mathcal{I}}\beta_j - \ln\beta_{i^\star}\right) < \epsilon_1 \quad \textit{with } \epsilon_1 < \epsilon. \tag{11}$$

Then, $p_t^{i^\star}$ governed by (7) satisfies $p_t^{i^\star} \to 1$ as $t \to \infty$. □

Proof. Define

$$L_t = \frac{p_t^j}{p_t^{i^\star}}; \quad j \in \mathcal{I}.$$

From (7) we have

$$p_t^i = p_0^i \prod_{\tau=0}^{t-1} \frac{\beta_i e^{-w_\tau^i}}{\sum_{j=1}^N p_\tau^j \beta_j e^{-w_\tau^j}},$$

from which it follows that

$$L_{t+n} = [\prod_{\tau=t}^{t+n-1} \frac{\beta_j e^{-w_\tau^j}}{\beta_{i^\star} e^{-w_\tau^{i^\star}}}]\, L_t. \tag{12}$$

Taking logarithms of both sides,

$$\begin{aligned} \ln\frac{L_{t+n}}{L_t} &= \sum_{\tau=t}^{t+n-1}\ln(\beta_j e^{-w_\tau^j}) - \sum_{\tau=t}^{t+n-1}\ln(\beta_{i^\star} e^{-w_\tau^{i^\star}}) \\ &\leq \sum_{\tau=t}^{t+n-1}\ln(\overline{\beta_j} e^{-w_\tau^j}) - \sum_{\tau=t}^{t+n-1}\ln(\beta_{i^\star} e^{-w_\tau^{i^\star}}) \\ &= n\ln\overline{\beta_j} - n\ln\beta_{i^\star} + \sum_{\tau=t}^{t+n-1} w_\tau^{i^\star} - \sum_{\tau=t}^{t+n-1} w_\tau^j, \end{aligned} \tag{13}$$

where $\overline{\beta_j} = \max_{s\in\mathcal{I}}\beta_s$.

From (13), (10), and (11) it can be concluded that there exists a positive γ such that

$$\ln\frac{L_{t+n}}{L_t} \leq -n\gamma \tag{14}$$

or, equivalently,

$$L_{t+n} \leq e^{\left(-n\gamma\right)} L_t. \tag{15}$$

It follows that $L_t = \frac{p_t^j}{p_t^{i^\star}}$ converges to zero for all $j \in \mathcal{I}$, as $n \to \infty$. Since $\sum_{i=1}^N p_0^i = 1$, it is now straightforward to conclude that $p_t^j \to 0$, and $p_t^{i^\star} \to 1$. Furthermore, the convergence is exponentially fast. □

Condition (10) can be viewed as a distinguishability criterion and implies that for sufficiently large t one of the local observers will exhibit least output error (residual) "energy". In fact, the following Corollary holds.

Corollary 1. *Suppose that the conditions of Theorem 2 hold and let the input signal $\eta = [\xi_t\ \theta_t\ u_t]^T$ consist of a bounded-spectral sequence with power spectral density $\Psi_\eta(\omega)$. Further, let $w_t^i := \frac{1}{2}\|y_t - \hat{y}_{t|i}\|^2_{S_{\kappa_i}^{-1}}$. Then, the* *parameter estimate $\hat{\kappa}_t$ converges to the closest to the true parameter κ as $t \to \infty$, in the following sense:*

$$\lim_{t\to\infty} \hat{\kappa}_t = \kappa_{i^\star}, \tag{16a}$$

$$i^\star = \arg \min_{i\in\{1,\ldots,N\}} \{\Upsilon_{\kappa_1,\kappa}, \Upsilon_{\kappa_2,\kappa}, \ldots, \Upsilon_{\kappa_N,\kappa}\} \tag{16b}$$

$$\Upsilon_{\kappa_i,\kappa} = tr[\frac{1}{2\pi}\int_{-\pi}^{\pi} (\mathbb{H}_i(e^{j\omega})\Psi_\eta(\omega)\mathbb{H}_i(e^{j\omega})^H S_{\kappa_i}^{-1})\, d\omega] \tag{16c}$$

where $\mathbb{H}_i(z)$ is the discrete transfer matrix defined by

$$\mathbb{H}_i(z) = \mathbb{C}_i(zI - \mathbb{A}_i)^{-1}\mathbb{B}_i + \mathbb{D}, \tag{17}$$

with

$$\mathbb{A}_i := \begin{bmatrix} A_\kappa & 0 \\ H_{\kappa_i}C_\kappa & A_{\kappa_i} - H_{\kappa_i}C_{\kappa_i} \end{bmatrix}, \quad \mathbb{C}_i := \begin{bmatrix} C_\kappa & -C_{\kappa_i} \end{bmatrix},$$
$$\mathbb{B}_i := \begin{bmatrix} L_\kappa & 0 & B_\kappa \\ 0 & H_{\kappa_i} & B_{\kappa_i} \end{bmatrix}, \qquad \mathbb{D} := \begin{bmatrix} 0 & I & 0 \end{bmatrix}.$$

A proof is available in [18].

Remark 2. In the stochastic MMAE, when the inputs to (17) are discrete-time white noise with intensity matrix Q, $\Upsilon_{\kappa_i,\kappa}$ in (16) can be computed as

$$\Upsilon_{\kappa_i,\kappa} = tr[(\mathbb{C}_i\Sigma_i\mathbb{C}_i^T + \Theta)\ S_{\kappa_i}^{-1}],$$

where Σ_i satisfies the Lyapunov equation

$$\Sigma_i = \mathbb{A}_i\Sigma_i\mathbb{A}_i^T + \mathbb{B}_iQ\mathbb{B}_i^T.$$

□

We now analyze the situation when the nominal values κ_i, for $i = 1 \ldots N$ do not include the true parameter κ. As Theorem 2 shows, as long as the distinguishability condition holds, one of the dynamic weights p^i governed by (7), say $p^{i^\star}$, converges to 1 and the rest converge to 0. In this case, the actual parameter is identified as $\kappa_{i^\star}$. Notice however that it cannot be concluded that the true value of κ is actually $\kappa_{i^\star}$. Nevertheless, in a well defined sense it can be said that the true value of κ is closer to $\kappa_{i^\star}$ than to any other κ_i for $i \in \mathcal{I}$. This simple reasoning allows us to conclude that, corresponding to each κ_i,; $i = 1 \ldots N$ there is a set of plants that are naturally identified as κ_i.

In what follows, we call each of these sets a set of Equivalently Identified Plants (EIP), denoted S^i_{EIP}. With an obvious abuse of notation, for each κ_i we define the corresponding EIP as a subset in the uncertain parameter space κ with the property that if the uncertain parameter belongs to that subset, then the selected model with parameter κ_i will be identified. Corollary 1 provides a method to compute the set of Equivalently Identified Plants (EIP) for all the κ_i.

Fig. 2 illustrates graphically how the EIP sets can be obtained. In Fig. 2, it is assumed that a scalar uncertain parameter κ lies in the interval $[\kappa_L\, , \, \kappa_U]$, and three local observers are designed based on κ_1, κ_2, and κ_3. We have plotted the weighted RMS of the output estimation error for each observer as a function of κ in $[\kappa_L\, , \, \kappa_U]$. The intersections of these curves define the boundaries of the EIP sets.

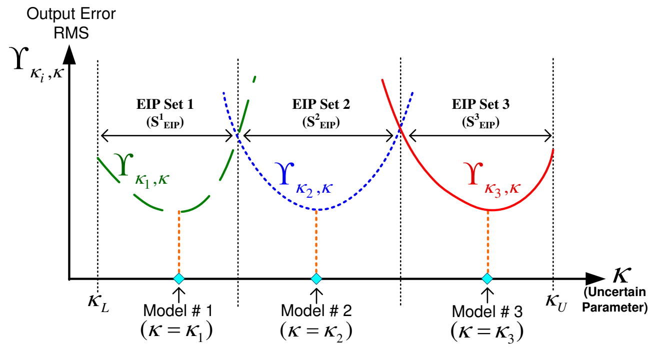

Fig. 2. Graphical Illustration of Equivalently Identified Plant (EIP) Sets and Models

Remark 3. A systematic methodology to compute EIP sets is as follows:
1. For each κ_i, $i = 1 \ldots N$ compute $\Upsilon_{\kappa_i,\kappa}$ for $\kappa \in [\kappa_L\, , \, \kappa_U]$.
2. The EIP set for i^{th} model is defined by

$$S^i_{EIP} = \{\kappa : \Upsilon_{\kappa_i,\kappa} = \min_{i\in\{1,\ldots,N\}} \{\Upsilon_{\kappa_1,\kappa}, \Upsilon_{\kappa_2,\kappa}, \ldots, \Upsilon_{\kappa_N,\kappa}\}\}.$$

□

IV. EFFECT OF UNMODELED DYNAMICS

Consider a discrete-time, multi-input, multi-output linear time-invariant plant with unmodeled dynamics described by multiplicative unstructured uncertainty of the type

$$G(z) = G_0(z)[I + W(z)\Delta(z)]. \tag{18}$$

where $G_0(z)$ is the nominal plant discrete transfer matrix, $W(z)$ is a known fixed stable discrete transfer matrix, and $\Delta(z)$ represents any unknown stable discrete transfer matrix satisfying $\|\Delta(z)\|_\infty \leq 1$. See Fig. 3, where $H(z)$ denotes

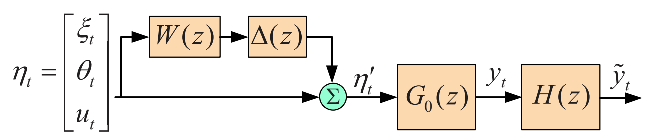

Fig. 3. Uncertain Plant (Nominal Discrete Transfer Matrix with Multiplicative Uncertainty) and Local Observer

the discrete transfer matrix of the local observer. The power spectral density of sequence η'_t in Fig. 3 is given by

$$\Psi_{\eta'}(\omega) = |I + W(e^{j\omega})\Delta(e^{j\omega})|^2\Psi_\eta(\omega).$$

Since $\Delta(z)$ is unknown and satisfies $\|\Delta(z)\|_\infty \leq 1$, the following holds:

$$[H_{LB}(e^{j\omega})]^2\Psi_\eta(\omega) \leq \Psi_{\eta'}(\omega) \leq [H_{UB}(e^{j\omega})]^2\Psi_\eta(\omega), \tag{19}$$

where

$$H_{UB}(e^{j\omega}) := \sup_{\|\Delta(e^{j\omega})\|_\infty \le 1} |I + W(e^{j\omega})\Delta(e^{j\omega})| = I + |W(e^{j\omega})|, \tag{20a}$$

$$H_{LB}(e^{j\omega}) := \inf_{\|\Delta(e^{j\omega})\|_\infty \le 1} |I + W(e^{j\omega})\Delta(e^{j\omega})| = \begin{cases} 0 & \text{if } |W(e^{j\omega})| \ge 1 \\ I - |W(e^{j\omega})| & \text{if } |W(e^{j\omega})| < 1 \end{cases} \tag{20b}$$

Using (16c) and substituting $\Psi_\eta(\omega)$ by the upper and lower bound of $\Psi_{\eta'}(\omega)$ determined in (19), one can compute an uncertainty band around the nominal RMS of the output error sequence. As shown in Fig. 4, the unmodeled dynamics lead

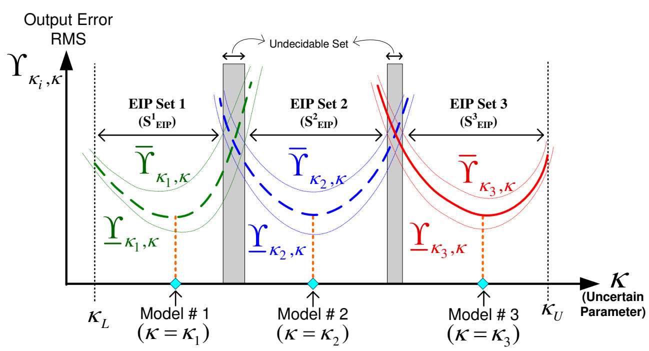

Fig. 4. Graphical Illustration of the effect of Unmodeled Dynamic on the Equivalently Identified Plant (EIP) Sets

to "undecidable sets" in the uncertain parameter space. If the true parameter lies in one of these sets, then it cannot be ascertained which model will be selected. Computer simulations have shown for specific examples that the larger the size of the unmodeled dynamics, the larger the undecidable set is. Notice, however that by exploiting the above circle of ideas one obtains a clear procedure to compute the undecidable sets.

Remark 4. The modified methodology to compute EIP sets in the presence of unmodelled dynamics is as follows:
1. For each κ_i, $i = 1 \dots N$ compute the upper and lower bounds of $\Upsilon_{\kappa_i,\kappa}$ for $\kappa \in [\kappa_L\,,\, \kappa_U]$ given by

$$\overline{\Upsilon}_{\kappa_i,\kappa} = tr[\frac{1}{2\pi}\int_{-\pi}^{\pi} (\mathbb{G}^i_{UB}(e^{j\omega})\Psi_\eta(\omega)\mathbb{G}^i_{UB}(e^{j\omega})^H S^{-1}_{\kappa_i})\, d\omega] \tag{21a}$$

$$\underline{\Upsilon}_{\kappa_i,\kappa} = tr[\frac{1}{2\pi}\int_{-\pi}^{\pi} (\mathbb{G}^i_{LB}(e^{j\omega})\Psi_\eta(\omega)\mathbb{G}^i_{LB}(e^{j\omega})^H S^{-1}_{\kappa_i})\, d\omega] \tag{21b}$$

where $\mathbb{G}^i_{LB}$ and $\mathbb{G}^i_{UB}$ are transfer matrices from input η to local estimation error $\tilde{y}_t$ (see Fig. 3) considering the upper and lower bounds of multiplicative uncertainty as computed in (20).
2. The EIP set for i^{th} model is defined by

$$S^i_{EIP} = \{\kappa : \overline{\Upsilon}_{\kappa_i,\kappa} \le \underline{\Upsilon}_{\kappa_j,\kappa} \quad \text{for all} \quad j \in \{1, 2, \dots, N\} \backslash i\}.$$

□

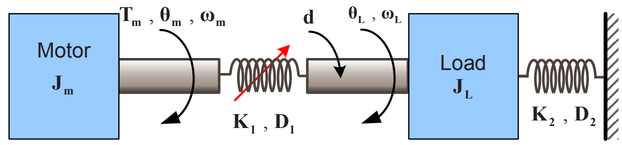

Fig. 5. Motor and Load with Elastic Transmission (the spring-constant K_1 is uncertain.)

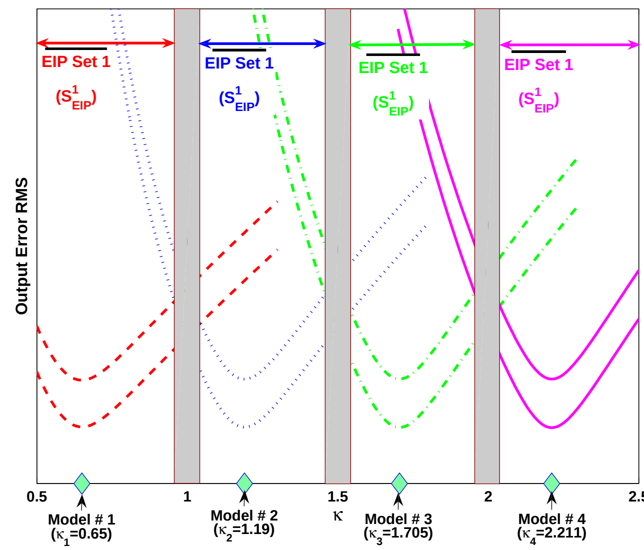

Fig. 6. Effect of Unstructured Uncertainty on Equivalently Identified Plant (EIP) Sets.

V. ILLUSTRATIVE EXAMPLE

The ME-MMAE procedure is now evaluated through the example depicted in Fig. 5. The plant consists of a motor driving a load through a flexible coupling and the load is connected to a wall trough a torsional spring and torsional damper. The output signal is the load shaft angle corrupted by measurement noise θ_t and the disturbance torque ξ_t affects only the load. The measuring functions $w^i_t = \frac{1}{2}(y_t - \hat{y}_{t|i})^T S^{-1}_{\kappa_i}(y_t - \hat{y}_{t|i})$ were used during the simulations. A state-space representation of the plant, including the disturbance and noise inputs, is given by (1) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{-K_1}{J_m} & \frac{K_1}{J_m} & \frac{-D_1}{J_m} & \frac{D_1}{J_m} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{K_1}{J_L} & \frac{-(K_1+K_2)}{J_L} & \frac{D_1}{J_L} & \frac{-(D_1+D_2)}{J_L} & \frac{1}{J_L} \\ 0 & 0 & 0 & 0 & -0.3 \end{bmatrix},\; B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{J_m} \\ 0 \\ 0 \end{bmatrix},$$

$$L' = [\,0\;0\;0\;0\;0.3\,], \quad C = [\,0\;0\;1\;0\;0\,],$$

where $J_m = J_L = 1\,(Kgm^2)$, $K_2 = 0.2\,(N/rad)$, $D_1 = D_2 = 0.1\,,(Ns/rad)$, and K_1 is an unknown parameter assumed to have a value in the interval $[0.5, \;\; 2.5]$; we divided uniformly the interval where the unknown parameter K_1 can live into 4 sub-intervals. We remark that the above dynamics are similar to the mass-spring-dashpot system (MSD) testbed for robust adaptive control (RMMAC), [14]. Using the results in Corollary 1, we computed the nominal values for K_1 such that the EIP set of each observer corresponds to the sub-intervals. Fig. 6 illustrates the procedure adopted with $N = 4$ local observers and the following set $\kappa = \{0.65, 1.19, 1.70, 2.21\}$ was obtained for nominal values. The y-axis corresponds to the weighted RMS of output error sequences. It is also assumed that input torque is provided through an amplifier whose bandwidth is unknown but in the interval $[20\ 25]$ rad/s; this amplifier can be described in the form of a model with multiplicative uncertainty with

$$G_0(z) = \frac{0.2015}{z - 0.7985}, \quad W(z) = \frac{0.1235z - 0.1235}{z - 0.8187}.$$

The undecidable sets between adjacent EIPs can be seen in Fig. 6.

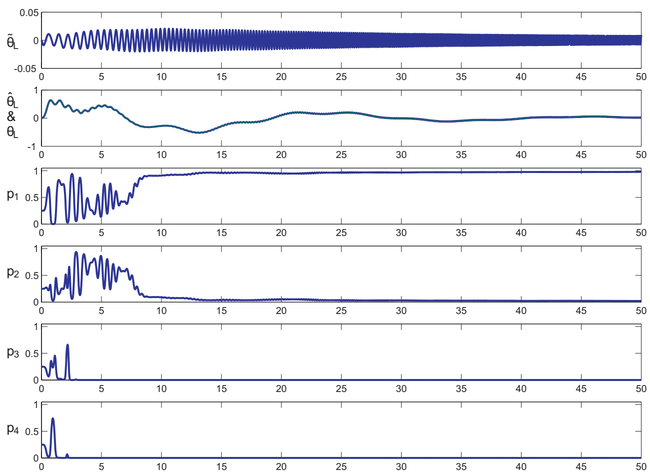

Fig. 7. Deterministic case: $K_1 = 0.92$.

Fig. 7 shows the time evolution of some representative signals. In this figure, the first sub-plot is the output estimation error and the second sub-plot is the output (load shaft angle) together with the estimated output (almost coincident). The remaining plots show the time evolution of the dynamic weighting signals. In the deterministic case, θ_t and ξ_t are swept-frequency cosine (chirp) sequences (with initial frequency 1 (Hz), target time 50 (sec), and final frequency 10 (Hz)). In Fig. 7, the unknown parameter K_1 is located near the boundary between two adjacent EIP sets, where identification is harder and the convergence takes longer. We have used different values of the K_1 in the simulations in both the stochastic and deterministic setups (in the stochastic case, θ_t and ξ_t are discrete-time white noise sequences) and except for the values which are very close to the boundaries, the correct model is always identified (the results are not shown due to the space limitations).

VI. CONCLUSIONS AND FUTURE RESEARCH

We presented and analyzed a class of Minimum Energy MMAE systems for LTI MIMO discrete systems with parametric uncertainty. We showed that if some suitable distinguishability condition holds, then the model identified is the SM that exhibits least output error "energy". Based on the energy of the output error sequences, we introduced the concept of Equivalently Identified Plant (EIP) sets corresponding to each local observer. The methodology proposed allowed us to deal with both parametric and unstructured uncertainty. In particular, we introduced the concept of undecidable sets, which capture the fact that unmodeled uncertainty will necessarily lead to basic limitations to identification. This issue and that of determining general geometric properties of the EIP sets deserve further research. Another topic that warrants consideration is that of deriving adaptive control systems for uncertain plants (especially unstable and non-minimum phase plants) using the multiple model approach.

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