# CYCLIC DIRECTED FORMATIONS OF MULTI-AGENT SYSTEMS

**Paulo Tabuada**[1] **George J. Pappas**[2] **Pedro Lima**[1]

[1] Instituto de Sistemas e Robótica
Instituto Superior Técnico
1049-001 Lisboa - Portugal
Fax: (351) 21 841 8291
{tabuada,pal}@isr.ist.utl.pt

[2] Department of EE
University of Pennsylvania
Philadelphia, PA 19104
Fax: (251) 573 2068
pappasg@seas.upenn.edu

**Keywords:** Multi-Agent systems, formations, cyclic graphs.

## Abstract

Formations of multi-agent systems, such as satellites, aircrafts and mobile robots require that individual agents satisfy their kinematic equations while constantly maintaining inter-agent constraints. In previous work we introduced the concept of *undirected formation graphs* and *directed formation graphs* to model such formations and presented conditions to determine formation feasibility. However the directed formations were only analyzed in the absence of *cycles* in the formation graph. In this paper we extend our previous results to accommodate also the presence of cycles in directed formations. Differential geometric and algebraic conditions are presented to determine feasibility of directed formations with possible cycles.

## 1 Introduction

Advances in communication and computation have enabled the distributed control of multi-agent systems. This philosophy has resulted in the next generation of automated highway systems [10], coordination of aircraft in future air traffic management systems [9], as well as formation flying aircraft, satellites, and multiple mobile robots [3, 4, 5, 6].

The control of multiple homogeneous or heterogeneous agents raises fundamental questions regarding the *formation control* of a group of agents. Multi-agent formations require individual agents to satisfy their kinematics while constantly satisfying inter-agent constraints. In typical leader-follower formations, the leader has the responsibility of guiding the group, while the followers have the responsibility of maintaining the inter-agent formation. Distributing the group control tasks to individual agents must be compatible with the control and sensing capabilities of the individual agents. As the inter-agent dependencies get more complex, a systematic framework for controlling formations is vital.

In our previous work [8] a framework for formation control of multi-agent systems was proposed. Formations were modeled using *formation graphs* which are graphs whose nodes capture the individual agent kinematics, and whose edges represent inter-agent constraints that must be satisfied. A similar approach has also been proposed in [6] and in [2] the coordination between agents is specified by a discrete set of way points instead of rigid inter agent constraints. We assume kinematic models for each agent described by drift free control systems. This class of systems is rich enough to capture holonomic, nonholonomic, or underactuated agents. The results presented in this paper extend the feasibility criteria for *directed formations* with possible *cycles*. These criteria are based on the concepts of *undirected formations* and *directed formations* that were also introduced in [8].

In this paper we propose a criteria to determine feasibility of directed formations with possible cycles. The cycles are analyzed individually and replaced by *macro-vertices* constituting an abstraction of the kinematics of the agents linked by the cycle. After all cycles have been replaced, the resulting acyclic formation graph can be analyzed by the methods described in [8]. The results are illustrated by analyzing a cyclic formation conceived to model 3 robots transporting a rigid object like a table or a box.

## 2 Feasible Formations

In this section we will review the concepts of undirected an directed formations as well as its feasibility characterizations that were introduced in [8]. We assume the reader is familiar with various differential geometric concepts at the level of [1].

Consider $n$ heterogeneous agents with states $x_i(t) \in M_i$, $i = 1, \ldots, n$ whose kinematics are defined by drift free controlled distributions on manifolds $M_i$ as:

$$\begin{aligned} \Delta_i &: M_i \times U_i \longrightarrow TM_i \\ \Delta_i &= \sum_j X_j u_j \end{aligned} \quad (1)$$

where $U_i$ is the control space, and the vector fields $X_j$ form a basis for the distribution. The controlled distributions are general enough to model nonholonomy and underactuation. A distribution $\Delta_i$ can be equivalently defined by its annihilating codistribution $\omega_{K_i}$ defined as [7]:

$$\omega_{K_i} = \{\alpha \in T^*M_i \quad | \quad \alpha(\Delta) = 0\} \tag{2}$$

The formation of a set of agents is defined by the *formation graph* which completely describes individual agent kinematics and global inter-agent constrains.

**Definition 2.1 (Formation Graph)** *A formation graph $F = (V, E, C)$ consists of:*

- *A finite set $V$ of vertices who's cardinality is equal to the number of agents. Each vertex $v_i : M_i \times U_i \longrightarrow TM_i$ is a distribution $\Delta_i$ modeling the kinematics of each individual agent as described in (1).*
- *A binary relation $E \subset V \times V$ representing a link between agents.*
- *A family of constraints $C$ indexed by the set $E$, $C = \{c_e\}_{e \in E}$. For each edge $e = (v_i, v_j)$, $c_e$ is a possibly time varying function $c_e(x_i, x_j, t) = 0$ describing the $\phi(e)$ independent constraints between vertices $v_i$ and $v_j$. For a generic edge $e = (v_i, v_j)$, $c_e$ is mathematically defined as $c_e : M_i \times M_j \times \mathbb{R} \longrightarrow \mathbb{R}^{\phi(e)}$, $\phi(e) \in \mathbb{N} \ \ \forall_{e \in E}$.*

Although our framework allows time-varying constraints, we shall assume time invariant constraints for the sake of clarity. We defer the reader to [8] for the full timed version. Two different types of formation graphs will be considered: undirected formations where $(V, E)$ will be an undirected graph and directed formations where $(V, E)$ will be a directed graph. In undirected formations, for each edge $e = (v_i, v_j)$ both agents are equally responsible for maintaining the associated constraint $c_e$, where as for directed formations the constraint $c_e$ must be maintained by agent $i$.

### 2.1 Undirected Formations

In undirected formations each agent is equally responsible for maintaining constraints. Because of this property it will be useful to collect all agent kinematics and constraints on a single manifold:

$$M = \prod_{i=1}^{n} M_i \tag{3}$$

Given an element $x$ of $M$ the canonical projection on the $i$th agent:

$$\pi_i : M \longrightarrow M_i \tag{4}$$

allow us to denote the state of the individual agents by $x_i = \pi_i(x)$. The formation kinematics is obtained by appending individual kinematics through direct sum, that is:

$$\begin{aligned} \Delta &: M \times U \longrightarrow TM \\ \Delta &= \oplus_{i=1}^{n} \Delta_i \end{aligned} \tag{5}$$

where $U$ is taken to be $U = \prod_{i=1}^{n} U_i$. To lift the individual constraints $c_e$ from $M_i \times M_j \times \mathbb{R}$, $i, j \in \{1, 2, \ldots, n\}$ to the group manifold $M$ we define $\mathcal{C}_e$ by:

$$\begin{aligned} \mathcal{C}_e &: M \times \mathbb{R} \longrightarrow \mathbb{R}^{\phi(e)} \\ \mathcal{C}_e(x, t) &= c_e(\pi_i(x), \pi_j(x), t) \end{aligned} \tag{6}$$

As explained in [8] all the relevant information regarding feasibility can be encoded in a single object. Consider an enumeration $\{1, 2, \ldots, m\}$ of the edges set $E$. Based on this enumeration we define the following vector valued form[1]:

$$\omega_F = \begin{bmatrix} \mathrm{d}\mathcal{C}_1 \\ \mathrm{d}\mathcal{C}_2 \\ \vdots \\ \mathrm{d}\mathcal{C}_m \end{bmatrix} \tag{7}$$

The kinematics can also be modeled as differential forms by constructing a vector valued form $\omega_K$ that annihilates control system (5) (see for *e.g.* [7]), that is:

$$\omega_K(X) = 0 \tag{8}$$

By combining the previous differential forms into the single object:

$$\Omega = \begin{bmatrix} \omega_F \\ \omega_K \end{bmatrix} \tag{9}$$

we can check for formation feasibility in a single equation as described in the next proposition:

**Proposition 2.2 ([8])** *If the formation constraints $C$ are time-invariant then the undirected formation is feasible iff $\Omega$ (thought as a pointwise linear map between vector spaces) is not of full rank.*

A solution of equation $\Omega(X) = 0$ specifies the motion of each individual agent. When more than one independent solution exists, a change in the direction of a single agent may require that all other agents also change their actions to maintain formation. This shows that, in general, solutions for undirected formations are centralized and require inter-agent communication for their implementation.

[1] This definition is independent of the chosen enumeration as can be easily verified.

### 2.2 Directed Formations

Another important class of formations can be modeled by directed graphs. A directed graph assigns responsibilities to the formation members in an asymmetric way. For each edge $e = (v_i, v_j)$ agent $i$ is responsible for maintaining the constraints $c_e$, while agent $j$ is not affected by the constraint of the edge.

Contrary to the undirected case were the symmetric distribution of responsibilities led to a single representation for the problem and its solutions, in the directed case the feasibility problem is naturally casted into a recursive procedure. This requires the following operators:

**Definition 2.3 (Post and Pre)** *Let $F = (V, E, C)$ be a directed formation graph. The* Post *operator is defined by*

$$\begin{array}{rcl} Post : V & \rightarrow & 2^V \\ v_i & \mapsto & \{v_j \in V \,:\, (v_i, v_j) \in E\} \end{array} \tag{10}$$

*Similarly, the* Pre *operator is defined as:*

$$\begin{array}{rcl} Pre : V & \rightarrow & 2^V \\ v_i & \mapsto & \{v_j \in V \,:\, (v_j, v_i) \in E\} \end{array} \tag{11}$$

Intuitively, $Post(v_i)$ will return the agents that are leading agent $i$, while $Pre(v_i)$ will return all the agents that are following agent $i$. $Post$ and $Pre$ extend to sets of vertices in the natural way, $Post(P) = \cup_{p \in P} Post(p)$ and $Pre(P) = \cup_{p \in P} Pre(p)$.

**Definition 2.4 (Leaders)** *A vertex $v_i$ is called a* leader *iff $Post(v_i) = \varnothing$.*

We will assume, for now, that a directed formation graph is a directed acyclic graph. In the next section we will see how cycles in formations can also be accommodated in the proposed framework.

We shall abuse notation a represent the distribution $\Delta_i$ defining the kinematics of agent $v_i$ by $\Delta(v_i)$ and for the set of agents $Post(v_i)$, $\Delta(Post(v_i)) = \oplus_{p \in Post(v_i)} \Delta(p)$ defined over $\Pi_{p \in Post(v_i)} M_p$. Similarly to the undirected case we define the following objects for each agent $i$:

$$\omega_F^i = \begin{bmatrix} \mathrm{d}c_1|_{x_j \text{ fixed}} \\ \mathrm{d}c_2|_{x_j \text{ fixed}} \\ \vdots \\ \mathrm{d}c_m|_{x_j \text{ fixed}} \end{bmatrix} \quad \omega_F^j = -\begin{bmatrix} \mathrm{d}c_1|_{x_i \text{ fixed}} \\ \mathrm{d}c_2|_{x_i \text{ fixed}} \\ \vdots \\ \mathrm{d}c_1|_{x_i \text{ fixed}} \end{bmatrix} \tag{12}$$

where $\{1, 2, \ldots m\}$ is an enumeration of the edges set between agent $i$ and its leaders ($Post(v_i)$). Similarly to the undirected case we define:

$$\Omega^i = \begin{bmatrix} \omega_F^i \\ \omega_K^i \end{bmatrix} \tag{13}$$

where $\omega_K^i$ is a vector valued form that annihilates agent $i$ kinematic distribution $\Delta(v_i)$. This motivates the following result analogous to the undirected case:

**Proposition 2.5 ([8])** *A directed formation is feasible iff the range of $\Omega^j|_{\Delta(Post(v_i))}$ is contained in the range of $\Omega^i$ for each agent $i$.*

Since Proposition 2.5 must be true for all agents, an algorithm can be constructed to determine feasibility.Let $L \subset V$ be a set of leaders and denote by $(\Omega^i)^{-1}(X)$ the set of preimages of $X$ under $\Omega^i$ and by $\mathcal{R}(S)$ the range of operator $S$.

**Algorithm 1 (Directed Feasibility)**

**initialization**: $V := L$
**while** $Pre(V) \neq \varnothing$ **do**
    $V := Pre(V)$
    **for all** $v_i \in V$ **do**
        $\Delta(v_i) := \mathbf{0}$
        **if** $\mathcal{R}(\Omega^j|_{\Delta(Post(v_i))}) \not\subseteq \mathcal{R}(\Omega^i)$
            **return UNFEASIBLE**
            **STOP**
        **else**
            $\Delta(v_i) := \Delta(v_i) + (\Omega^i)^{-1}(\mathcal{R}(\Omega^j|_{\Delta(Post(v_i))}))$
        **end if**
    **end**
**end**

**Theorem 2.6 ([8])** *Let $F = (V, E, C)$ be an acyclic, directed formation graph. Algorithm 1 terminates in a finite number of steps and returns:*

- *Unfeasible if the formation is not feasible.*
- *A distribution per agent specifying the available directions to maintain formation if the formation is feasible.*

## 3 Cyclic Directed Formations

To determine feasibility of directed formations with cycles, we analyze each cycle individually to determine its feasibility. In case all cycles are feasible they are replaced by macro vertices, thereby transforming a directed cyclic formations into an acyclic one. We start by considering a directed formation consisting of a single cycle. We propose a concept of solution and give conditions to determine feasibility of this formation. When there are several cycles in a formation we analyze each cycle individually and if solutions exist we replace it by its abstraction that we consider as a macro vertex. This procedure transforms a cyclic directed formation into an acyclic one where the methods described in the previous section can be applied to determine feasibility of the resulting acyclic formation.

### 3.1 Feasibility of Cycles

Determining a concept of solution for a directed cycle is not a simple task since the cyclic nature of the graph prevent us from using the concepts introduced for acyclic graphs. A solution must respect the distribution of roles dictated by the arrows in the graph, however it is not clear to say that agent 1 is the only responsible for the constraint between 1 and 2 since 2 may depend on 3 and 3 on 1. To set the ideas consider a cycle with three agents as pictured in 1.

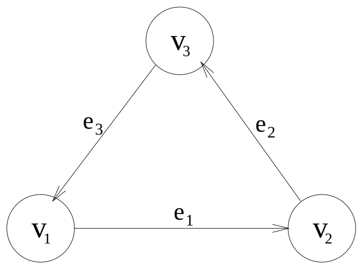


Figure 1: Graph associated with a directed formation consisting of cycle with three agents.

The first requirement that a concept of solution must satisfy is to be a (not necessarily proper) subset of the set of solutions of its undirected counterpart graph. Clearly if no undirected solutions exist, there are also no directed ones. The second characteristic of a solution of a cycle is based on the following observation. Suppose that agent 1 flows along direction $Y_{11}$ and responding to that, agent 3 maintains the constraint associated with edge $e_3$ by flowing along $Y_{32}$. Agent 2, responsible for constraint associated with edge $e_2$ chooses to flow along $Y_{22}$ and finally agent 1 to maintain the constraint that links it with agent 2 is forced to flow along a direction different from the initial one. This process of changing vector fields depending on the local leaders may repeat undefinitively since there is no cooperation between agents to negotiate their behavior coherently. Ruling out this kind of situations naturally leads to the following definition:

**Definition 3.1** *Let $F$ be a directed formation graph consisting of a single cycle. The formation defined by $F$ is feasible iff it is feasible as an undirected formation and for every agent $i$ in the formation the following must hold:*

*Let $X^u$ be an undirected solution of $F$ and let $X^e$ be a solution of the directed graph $F_e$ obtained from $F$ by removing the outgoing arrow $e$ from vertex $i$. If $\pi_i(X^u) = \pi_i(X^e)$ then $X^e$ must be an undirected solution of $F$. If these conditions are met, the solutions of $F$ are then given by $\bigcup_{e\in E} X^e$.*

This definition rules out the pathological situations previously described and admits the following simpler from. Consider an enumeration $\{e_1, e_2, \ldots, e_n\}$ of the edges set such that $e_i = (v_i, v_{i+1})$ for $i$ mod $n$ and let $S$ denote the set of undirected solutions of the formation $F$. Denote by $S_{e_i}$ the set of solutions of the formation $F_{e_i}$ obtained from $F$ by removing the edge $e_i$ and satisfying $\pi_{e_i}(S_{e_i}) = \pi_{e_i}(S)$. If $F$ is feasible we must have $S_{e_i} \subseteq S$ for every $e_i \in E$. Although we have provided a characterization of the feasibility of a cycle, this definition requires solving the undirected counterpart of the cycle as well as all the directed acyclic formations induced by the subgraphs with $n-1$ edges and $n$ vertices. A more convenient way to determine feasibility is given in the next result.

**Proposition 3.2** *Let $F$ be a directed formation graph consisting of a single cycle of $n$ agents. $F$ is feasible if $\mathrm{d}c_{e_i}|_{x_i \text{ fixed}} = -\mathrm{d}c_{e_i}|_{x_{i+1} \text{ fixed}}$ for all $i$ mod $n$.*

The proof of the above result requires the following standard lemma that we state without proof:

**Lemma 3.3** *Let $\omega_1$ and $\omega_2$ be two constant-rank codistributions on a smooth manifold $M$. Denote by $\Delta_i$ the subbundle of $TM$ annihilated by $\omega_i$. We have the inclusion $\Delta_1 \subseteq \Delta_2$ iff $\omega_2 \subseteq \omega_1$.*

Lets return to the proof of Proposition 3.2.

**Proof:** Feasibility of the formation $F$ is by definition equivalent to $S_{e_i} \subseteq S$ for all $e_i \in E$. The set of solutions $S_{e_i}$ is equivalently described by its annihilating codistribution $\omega^{e_i}$ given by:

$$\begin{aligned}\omega^{e_i} &= Span\{\mathrm{d}c_{e_1}, \mathrm{d}c_{e_2}, \ldots, \mathrm{d}c_{e_{i-1}}, \mathrm{d}c_{e_{i+1}}, \ldots, \mathrm{d}c_{e_n}\}\\ &+ Span\{\omega_K^1, \omega_K^2, \ldots, \omega_K^n\}\\ &+ Span(\pi_i^* \omega^{iS}) \qquad (14)\end{aligned}$$

where $\omega^{iS}$ is the codistribution on $M_i$ annihilating $\pi_i(X^u)$. By lemma 3.3 we have that

$$\{\mathrm{d}c_{e_1}, \mathrm{d}c_{e_2}, \ldots, \mathrm{d}c_{e_n}, \omega_K^1, \omega_K^2, \ldots, \omega_F^n\} \subseteq \omega^{e_i} \qquad (15)$$

and by construction of $\omega^{e_i}$ the last inclusion reduces to $\mathrm{d}c_{e_i} \in Span(\omega^{e_i})$. If the condition $\mathrm{d}c_{e_i}|_{x_i \text{ fixed}} = -\mathrm{d}c_{e_i}|_{x_{i+1} \text{ fixed}}$ for all $i$ mod $n$ holds straight forward computations show that $\mathrm{d}c_{e_i} \in Span(\mathrm{d}c_{e_j})$, $i \neq j$ and the result is proved. ■

Proposition 3.2 provides a easily checkable sufficient condition to determine the cycle feasibility. However under those conditions we have the following result relating the solutions of the cycle formation with an acyclic one.

**Proposition 3.4** *Let $F$ be a directed formation graph consisting of a single cycle of $n$ agents. If $\mathrm{d}c_{e_i}|_{x_i \text{ fixed}} = -\mathrm{d}c_{e_i}|_{x_{i+1} \text{ fixed}}$ for all $i$ mod $n$ holds then the solutions of $F$ can also be obtained by removing any of the formation constraints between the agents.*

**Proof:** From the proof of Proposition 3.2 we see that the condition $\mathrm{d}c_{e_i}|_{x_i \text{ fixed}} = -\mathrm{d}c_{e_i}|_{x_{i+1} \text{ fixed}}$ for all $i$ mod $n$ implies that $dc_{e_i} \in Span(dc_{e_j})$, $i \neq j$ meaning that any of the formation constraints can be removed without altering the vector space spanned by $\{dc_{e_1}, \ldots, dc_{e_n}\}$. In particular we have that $S_{e_i} = S_{e_j}$ and therefore $S_{e_j} = \bigcup_{e \in E} S_e$ for any $j = 1, \ldots, n$. We have then that the solutions $S_{e_i}$ for any fixed $i \in \{1, \ldots, n\}$ equal the solutions of the formation $F$. ■

This proposition shows how the feasibility conditions for a directed formation consisting of a single cycle are extremely tight and suggest that the modeling power offered by them is somewhat reduced.

### 3.2 Cycles and Macro Vertices

When the directed cycle results in a feasible formation, the solution space of the cycle provides an abstraction of the kinematics of the agents connected by the cycle. This solution space can be determined by the methodology described in Section 2. To compute the abstraction of macro-vertex one determines a basis $\{K_1, K_2, \ldots, K_k\}$ for the solution space. The basis vectors define a controlled distribution on $M_1 \times M_2 \times \ldots \times M_m$ by the expression $\sum_{i=1}^{k} K_i u_i$. The new formation graph is therefore obtained from $F = (V, E, C)$ by introducing the equivalence relation:

$$\begin{aligned} & R \subseteq V \times V \qquad (16) \\ & (v_i, v_j) \in R \text{ iff both } v_i \text{ and } v_j \text{ belong to the cycle} \end{aligned}$$

The quotient formation graph $F/R = (V/R, E/R, C/R)$ can be described by identifying all the vertices in $V$ that belong to the cycle. The representant of the cycle equivalence class is the macro-vertex

$$\begin{aligned} v : N \times V &\rightarrow TN \\ (y, u) &\mapsto \sum_{i=1}^{k} K(y)_i u_i \end{aligned} \qquad (17)$$

where $N = M_1 \times M_2 \times \ldots \times M_m$ and $V = U_1 \times U_2 \times \ldots \times U_m$. The new edges set $E/R$ is obtained from $E$ by replacing all pairs $(v_i, v_j) \in E$ such that $v_i$ or $v_j$ belong to the cycle by $(v, v_j)$ or $(v_i, v)$, respectively and eliminating all the edges defining the cycle. The family of constraints $C/R$ is given by all the constraints in $C$ now associated with edges in $E/R$.

In general given a directed formation with cycles, if all the cycles are feasible, they can be replaced by macro vertices and the remaining acyclic directed formation can be analyzed by the algorithm described in Section 2.

## 4 Example

To illustrate the proposed method to analyze cyclic directed formations we will consider a team of 3 robots transporting a rigid object like a table for example. We will assume a formation as represented in Figure 1 and consider three nonholonomic robots with kinematics given by:

$$X_i = \begin{bmatrix} \cos\theta_i \\ \sin\theta_i \\ 0 \end{bmatrix} u_1^i + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2^i \qquad (18)$$

for $i = 1, 2, 3$ on manifolds $M_i = \mathbb{R}^2 \times \mathbb{S}^1$. Since the table is a rigid object the formation must also behave as a rigid object in order not to drop the object. The natural constraints to associate with each edge of the formation are:

$$c_1 = \begin{bmatrix} x_1 - x_2 - d_1^x \\ y_1 - y_2 - d_1^y \end{bmatrix} \quad c_2 = \begin{bmatrix} x_2 - x_3 - d_2^x \\ y_2 - y_3 - d_2^y \end{bmatrix}$$
$$c_3 = \begin{bmatrix} x_3 - x_1 - d_3^x \\ y_3 - y_1 - d_3^y \end{bmatrix} \qquad (19)$$

where $d_1^x$, $d_1^y$, $d_2^x$, $d_2^y$, $d_3^x$ and $d_3^y$ are positive constants representing the distances (in the $x$ and $y$ directions) to be maintained between the robots. To analyze feasibility of the cycle we need to compute the following differential forms:

$$\begin{aligned} dc_1|_{x_2 \text{ fixed}} &= \begin{bmatrix} dx_1 \\ dy_1 \end{bmatrix} = -dc_3|_{x_3 \text{ fixed}} \\ dc_2|_{x_3 \text{ fixed}} &= \begin{bmatrix} dx_2 \\ dy_2 \end{bmatrix} = -dc_1|_{x_1 \text{ fixed}} \\ dc_3|_{x_1 \text{ fixed}} &= \begin{bmatrix} dx_3 \\ dy_3 \end{bmatrix} = -dc_2|_{x_2 \text{ fixed}} \end{aligned} \qquad (20)$$

The conditions of Proposition 3.2 are clearly satisfied and the cycle is feasible. To determine the abstraction of this cycle one computes the vector valued form $\Omega$ to obtain:

$$\Omega = \begin{bmatrix} dx_1 \\ dy_1 \\ dx_2 \\ dy_2 \\ dx_3 \\ dy_3 \\ \sin\theta_1 dx_1 - \cos\theta_1 dy_1 \\ \sin\theta_2 dx_2 - \cos\theta_2 dy_2 \\ \sin\theta_3 dx_3 - \cos\theta_3 dy_3 \end{bmatrix} \qquad (21)$$

The corresponding kernel is generated by the vectors:

$$K_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad K_2 = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \\ \cos\theta \\ \sin\theta \\ 0 \\ \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \tag{22}$$

with $\theta = \theta_1 = \theta_2 = \theta_3$. The abstracting mega vertex $v$ is a now control system on $M = M_1 \times M_2 \times M_3$ defined by $X = K_1 u_1 + K_2 u_2$. Proposition 3.4 tell us that we can remove any of the constraints without altering the solutions of the formation. This is a consequence of the tight conditions given by Proposition 3.2. To illustrate this fact it is worth to realize that the constraints:

$$\begin{aligned} c_1 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 - (d_1^x)^2 - (d_1^y)^2 \\ c_2 &= (x_2 - x_3)^2 + (y_2 - y_3)^2 - (d_2^x)^2 - (d_2^y)^2 \\ c_3 &= (x_3 - x_1)^2 + (y_3 - y_1)^2 - (d_3^x)^2 - (d_3^y)^2 \end{aligned} \tag{23}$$

cannot be used to specify this cyclic formation. If one removes the edge $e_3$ a possible configuration for the resulting directed formation is displayed in Figure 2 which is not a solution of $F$ if considered as an undirected formation since $v_i$ and $v_3$ no longer respect $c_3$. However both constraints (19) as well as constraints (23) produce the same solutions for $F$ as an undirected formation.

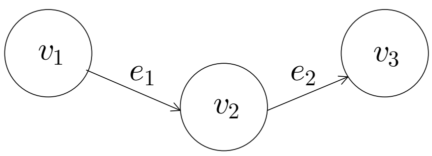


Figure 2: Possible configuration of the formation obtained from $F$ by removing the constraint associated with edge $e_3$.

## 5 Conclusions

This paper has extended our previous results on feasibility of directed formations by addressing the existence of cycles in the formation. The feasibility of such formations has been characterized in terms of the kernels of the constraints associated with the cycle edges. However the obtained conditions are very tight and imply the existence of acyclic formations with the same solutions. This result motivates the need for a better understanding of the relation between directed and undirected formations.

## 6 Acknowledgments


This work was performed while the first author was visiting the University of Pennsylvania. This research is partially supported by DARPA under grant N66001-99-C-8510, DARPA Grant F33615-00-C-1707, the University of Pennsylvania Research Foundation, and by Fundação para a Ciência e Tecnologia under grant PRAXIS XXI/BD/18149/98.


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