Output-feedback control for stabilization on $\mathrm{SE}(3)$

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***Abstract*— This paper addresses the problem of stabilizing systems that evolve on $\mathrm{SE}(3)$. The proposed solution consists of an output-feedback controller that guarantees almost global asymptotic stability (GAS) of the desired equilibrium point, i.e. the point is stable and, except for a set of zero measure, all initial conditions converge to it. The output vector is formed by the position coordinates, expressed in the body frame, of a collection of landmarks fixed in the environment. The resulting closed-loop system exhibits the following properties: i) the position error is globally exponentially stable and ii) the norm of the angle-axis of the error rotation matrix is monotonically decreasing almost everywhere. Results are also provided that allow one to select landmark configurations so as to control how the position and orientation of the rigid body converge to their desired values.**

I. INTRODUCTION

The problem of stabilizing a rigid-body in position and orientation is by no means a new control problem. Considering the simplest case of a fully-actuated kinematic model, the classical approach relies on a local parameterization of the rotation matrix, such as the Euler angles, which transforms the state-space into an Euclidean vector space. In this setting, the problem admits a trivial solution. However, no global solution can be obtained and there is no guarantee that the generated trajectories will not lead the system to one of its geometric singularities. Moreover, the described trajectories may be practically inadequate, since the module of the Euler angles vector does not correspond to a metric on $\mathrm{SO}(3)$. An alternative way of parameterizing rotations, which still has ambiguities but is globally nonsingular, is offered by the unit quaternions or the angle-axis parameterization. In these cases, global results can be obtained - see [1] for an example based on quaternions that solves an attitude regulation problem for low-Earth orbit rigid satellites and [2] for an example that uses the angle-axis representation to tackle a visual-servoing problem. However, both methods have the drawback of requiring full state knowledge and mapping the orientation to the selected parametrization.

In this paper, we present an output-feedback solution to the stabilization problem, defined on a setup of practical significance. It is assumed that there is a collection of landmarks fixed in the environment and that the coordinates of the landmarks' positions are provided in the body frame. This type of measurements are produced by a number of onboard sensors, including CCD cameras, ladars, pseudo-GPS, etc.

The main contribution of this paper is the design of an output-feedback control law, based on the above described measurements, that guarantees almost global asymptotic stability (GAS) of the desired equilibrium point. In loose terms, this corresponds to saying that the point is stable and, except for a zero measure set of initial conditions, the system converges asymptotically to that point [3]. The relaxation in the concept of GAS from global to almost global provides a suitable framework for the stability analysis of systems evolving on manifolds not diffeomorphic to an Euclidean vector space, as is the case of the Special Euclidean Group $\mathrm{SE}(3)$ [4]. As discussed in [5], [6], and [7], topological obstacles preclude the possibility of globally stabilizing these systems by means of continuous state feedback. The approach followed in this paper is in line with the methods presented in [5] and [4], which address the attitude tracking problem on $\mathrm{SO}(3)$ based on the so-called modified trace function. Building on these results, we address the more general problem of stabilization on $\mathrm{SE}(3)$ and, equally important, we provide a controller that only requires output feedback, as opposed to full-state. In addition, we establish results that describe the effect of the geometry of the points on the shape of the equilibria set and on the dynamic behaviour of the closed-loop system. Namely, using the angle-axis parameterization for the error rotation matrix, the decreasing monotonicity the rotation angle's absolute value is ensured almost everywhere and it is shown that almost GAS of the axis of rotation at a given point can be obtained by appropriate landmark placement.

The paper is organized as follows. Section II introduces the problem of stabilization on $\mathrm{SE}(3)$ and defines the output vector considered. Section III describes the construction of an almost globally asymptotically stabilizing state feedback controller for the system at hand. In the process, an exact expression for the region of attraction is derived. In Section III-A, we show that the proposed control law can be expressed solely in terms of the output, and then analyze the convergence of the position error and of the angle and axis of rotation arising from the angle-axis parameterization of the error rotation matrix. Simulation results that illustrate the performance of the control system are presented in Section IV. Section V summarizes the contents of the paper and presents directions for future work. For the sake of brevity, most of the proofs and technical results are omitted

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from the paper, and the reader is referred to [8] for a comprehensive presentation of this material.

II. Problem Formulation

Consider a fully-actuated rigid-body, attached to a frame $\{b\}$ and whose kinematic model is described by

$$\dot{\mathbf{p}} = -\mathbf{v} - S(\boldsymbol{\omega})\mathbf{p} \tag{1a}$$

$$\dot{R} = -S(\boldsymbol{\omega})R, \tag{1b}$$

where $(\mathbf{p}, R) = \left({}^{b}\mathbf{p}_{\pi},\ {}^{b}_{\pi}R\right) \in \mathrm{SE}(3)$ denotes the configuration of a fixed frame $\{\pi\}$ with respect to $\{b\}$, $\mathbf{v}$, $\boldsymbol{\omega} \in \mathbb{R}^3$ the linear and angular velocities of $\{b\}$ with respect to $\{\pi\}$, expressed in $\{b\}$, and $S(.)$ is a function from $\mathbb{R}^3$ to the space of three by three skew-symmetric matrices $\mathcal{S} = \{M \in \mathbb{R}^{3\times 3} : M = -M^T\}$ defined by

$$S\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}. \tag{2}$$

Note that S is a bijection and verifies $S(\mathbf{a})\mathbf{b} = \mathbf{a} \times \mathbf{b}$, where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ and $\times$ is the vector cross product.

Consider also a target configuration $(\mathbf{p}^*, R^*) = \left({}^{d}\mathbf{p}_{\pi},\ {}^{d}_{\pi}R\right) \in \mathrm{SE}(3)$, defined as the configuration of $\{\pi\}$ with respect to the desired body frame $\{d\}$, which is assumed to be fixed in the workspace. Fig. 1 illustrates the setup at hand, where the coordinates of n points acquired at the current and desired configurations $(\mathbf{p}, R)$ and $(\mathbf{p}^*, R^*)$, respectively, are available to the system for feedback control. In loose terms, the control objective consists of designing a control law for $\mathbf{v}$ and $\boldsymbol{\omega}$, based on the available current and desired point coordinates, which ensures the convergence of $(\mathbf{p}, R)$ to $(\mathbf{p}^*, R^*)$ (or, equivalently, of $\{b\}$ to $\{d\}$), with the largest possible basin of attraction.

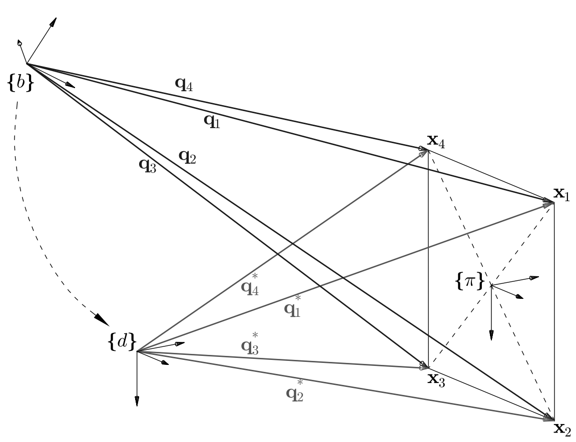


Fig. 1. Problem setup.

The landmarks, whose position coordinates in $\{\pi\}$ are denoted by $\mathbf{x}_j \in \mathbb{R}^3$, $j \in \{1, 2, \ldots, n\}$, are required to satisfy the following conditions:

Assumption 1: At least three of the $\mathbf{x}_j$ are not collinear.

Assumption 2: The origin of $\{\pi\}$ coincides with the centroid of the landmark points, such that $\sum_j \mathbf{x}_j = 0$.

Choosing this placement for $\{\pi\}$ considerably simplifies the forthcoming derivations and implies no loss of generality. As shown latter in the paper, Assumption 1 can be interpreted as an observability condition.

To conclude the problem formulation, we introduce the error variables

$$\mathbf{e} = \mathbf{p} - \mathbf{p}^* \in \mathbb{R}^3, \quad R_e = R^{*T} R \in \mathrm{SO}(3), \tag{3}$$

and the output vector

$$\mathbf{y} = [\mathbf{q}_1^T \ \ldots \ \mathbf{q}_n^T]^T \in \mathbb{R}^{3n\times 1}, \tag{4}$$

where $\mathbf{q}_j = R\mathbf{x}_j + \mathbf{p}$, $\ j \in \{1, 2, \ldots, n\}$, denotes the coordinates of the jth point expressed in $\{b\}$. Similarly, we define the desired output vector $\mathbf{y}^* = [\mathbf{q}_1^{*T} \ \ldots \ \mathbf{q}_n^{*T}]^T \in \mathbb{R}^{3n\times 1}$, where $\mathbf{q}_j^* = R^*\mathbf{x}_j + \mathbf{p}^*$. From a practical point of view, (4) should be viewed as the output vector. Note that $\mathbf{q}_j$ and $\mathbf{q}_j^*$ are precisely the type of measurements produced by on-board sensors that are able to locate landmarks fixed in the environment. As on-board sensors, they produce the coordinates of the landmarks' positions in the body frame. Examples of such sensors include CCD cameras, ladars, pseudo-GPS, etc.

The state-space model for the error system can be written as

$$\dot{\mathbf{e}} = -\mathbf{v} - S(\boldsymbol{\omega})(\mathbf{e} + \mathbf{p}^*) \tag{5a}$$

$$\dot{R}_e = -S(R^{*T}\boldsymbol{\omega})R_e, \tag{5b}$$

with the output vector given by (4). The control objective can then be defined as that of designing a control law based on $\mathbf{y}$ that drives $\mathbf{e}$ to zero and R_e to the identity matrix I_3.

III. Control design on SE(3)

The approach adopted to solve the proposed stabilization problem builds on Lyapunov theory and, for that purpose, the following candidate Lyapunov function is considered

$$V = \frac{1}{2}\|\mathbf{y} - \mathbf{y}^*\|^2 = \frac{1}{2}\sum_j \left\|\mathbf{q}_j - \mathbf{q}_j^*\right\|^2. \tag{6}$$

Since we are concerned with the global asymptotic stabilization (GAS) of a system evolving on SE(3), it is convenient to express V as a function on SE(3). As shown in [8], V can be written as

$$V(\mathbf{e}, R_e) = V_1(\mathbf{e}) + V_2(R_e), \tag{7}$$

where

$$V_1(\mathbf{e}) = \frac{n}{2}\mathbf{e}^T\mathbf{e}, \tag{8}$$

$$V_2(R_e) = \mathrm{tr}\left((I - R_e)XX^T\right), \tag{9}$$

and

$$X = \begin{bmatrix}\mathbf{x}_1 \ \ldots \ \mathbf{x}_n\end{bmatrix} \in \mathbb{R}^{3\times n}.$$

Using the angle-axis representation for rotations, such that $R_e = \mathrm{rot}(\theta, \mathbf{n}) = I_3 + \sin\theta \ S(\mathbf{n}) + (1 - \cos\theta)S(\mathbf{n})^2$ represents a rotation of angle $\theta \in [0, \pi]$ about the axis $\mathbf{n} \in \mathbb{S}^2$, one can show that (9) can be rewritten as

$$V_2(R_e) = (1 - \cos\theta)\mathbf{n}^T P\mathbf{n}, \tag{10}$$

where

$$P = \mathrm{tr}(XX^T)I_3 - XX^T.$$

Using expressions (8)-(9), it is straightforward to show that, as long as three landmark points are noncollinear, V satisfies the condition $V = 0$ if and only if $\mathbf{e} = 0$ and $R_e = I_3$. The time derivatives of V_1 and V_2 take the form

$$\dot{V}_1 = -n\mathbf{e}^T(\mathbf{v} - S(\mathbf{p}^*)\boldsymbol{\omega}) \quad (11)$$

$$\dot{V}_2 = -S^{-T}(R_eXX^T - XX^TR_e^T)R^{*T}\boldsymbol{\omega}, \quad (12)$$

respectively, yielding

$$\dot{V} = -\mathbf{a}_v^T\mathbf{v} - \mathbf{a}_\omega^T\boldsymbol{\omega}, \quad (13)$$

where $\mathbf{a}_v = n\mathbf{e}$, $\mathbf{a}_\omega = nS(\mathbf{p}^*)\mathbf{e} + R^*S^{-1}(R_eXX^T - XX^TR_e^T)$, and $S^{-1} : \mathcal{S} \mapsto \mathbb{R}^3$ corresponds to the inverse of the skew map S defined in (2). Once again, using the angle-axis representation of R_e, the derivative of V_2 can be rewritten as

$$\dot{V}_2 = -\mathbf{n}^TPQ(\theta,\mathbf{n})^TR^{*T}\boldsymbol{\omega}, \quad (14)$$

where $Q(\theta,\mathbf{n}) = \sin\theta I_3 + (1-\cos\theta)S(\mathbf{n})$. Details on the derivation of the expressions presented for V_1, V_2, and respective derivatives can be found in [8].

Before presenting a possible solution to the stabilization problem, we describe a preliminary approach to the problem that serves as motivation. Given (13), the simplest state-feedback control law yielding $\dot{V} \le 0$ would be

$$\mathbf{v} = k_v\mathbf{a}_v, \;\; \boldsymbol{\omega} = k_\omega\mathbf{a}_\omega, \quad (15)$$

where $k_v > 0$ and $k_\omega > 0$. This choice of controller guarantees, by Lyapunov's stability theorem, local stability of $(\mathbf{e}, R_e) = (0, I_3)$ and, by LaSalle's theorem, global convergence to the largest invariant set in the domain satisfying $\dot{V} = 0$. In this particular case, the whole set defined by $\dot{V} = 0$ is positively invariant, since all its elements are equilibrium points of the system. In summary, GAS would only be guaranteed if $(\mathbf{e}, R_e) = (0, I_3)$ were the unique solution of $\dot{V} = 0$. The following result discards this possibility.

Lemma 3.1: ([8]) Under Assumptions 1 and 2, the derivative of V along trajectories of the system (5) with the control law (15) is equal to zero if and only if $\mathbf{e} = 0$ and R_e belongs to the set

$$\mathcal{C}_{V_2} = I_3 \cup \{\mathrm{rot}(\pi,\mathbf{n}_i) \in \mathrm{SO}(3) : \mathbf{n}_i \text{ is an eigenvector of } P\}. \quad (16)$$

In addition, P can be factored as $U\Lambda U'$ with $U \in \mathrm{O}(3)$ and $\Lambda = \mathrm{diag}(\sigma_2^2+\sigma_3^2, \sigma_1^2+\sigma_3^2, \sigma_1^2+\sigma_2^2)$, where $\sigma_1 \ge \sigma_2 \ge \sigma_3 > 0$ are the singular values of X. Three cases may occur:

i) if all singular values of X are distinct, then $\mathcal{C}_{V_2}$ is given by

$$\mathcal{C}_{V_2} = \{I_3\} \cup \{\mathrm{rot}(\pi,\mathbf{n}_j) : j = 1,2,3\};$$

ii) if only two are distinct, then $\mathcal{C}_{V_2}$ consists of the larger set

$$\mathcal{C}_{V_2} = \{I_3\} \cup \{\mathrm{rot}(\pi,\mathbf{n}) : \mathbf{n} = \mathbf{n}_k \text{ or } \mathbf{n} \in \mathrm{span}(\mathbf{n}_i,\mathbf{n}_j) \cap \mathbb{S}^2, \sigma_i = \sigma_j \ne \sigma_k,\, i,j,k = 1,2,3\};$$

iii) otherwise (all singular values of X equal), $\mathcal{C}_{V_2}$ is given by

$$\mathcal{C}_{V_2} = \{I_3\} \cup \{\mathrm{rot}(\pi,\mathbf{n}) : \mathbf{n} \in \mathbb{S}^2\}.$$

Since the matrix P is completely determined by the point positions that define X, Lemma 3.1 shows that C_{V_2} is completely determined by the geometry of the measured points, which, in many applications, can be placed to yield appropriate sets C_{V_2}. To illustrate this observation, consider two configurations for the landmark points, a rectangle and a square, corresponding to the matrices $X_1 = \begin{bmatrix} a & a & -a & -a \\ b & -b & -b & b \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $X_2 = \begin{bmatrix} a & a & -a & -a \\ a & -a & -a & a \\ 0 & 0 & 0 & 0 \end{bmatrix}$ respectively, with $a > b > 0$. It is easy to show that, in the first case, V_2 has exactly four critical points given by $C_{V_2}^{(1)} = \{I_3, \mathrm{diag}(-1,-1,1)\} \cup \{\mathrm{diag}(-1,1,-1), \mathrm{diag}(1,-1,-1)\}$, while, in the second, the critical points of V_2 form the connected set

$$C_{V_2}^{(2)} = \{I_3, \mathrm{diag}(-1,-1,1)\} \cup \{R_e \in \mathrm{SO}(3) : R_e = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ \sin\psi & -\cos\psi & 0 \\ 0 & 0 & -1 \end{bmatrix}, \psi \in \mathbb{R}\}. \quad (17)$$

The sets $C_V^{(1)} = \{0\} \times C_{V_2}^{(1)}$ and $C_V^{(2)} = \{0\} \times C_{V_2}^{(2)}$ are depicted in Fig. 2(a) and (b), respectively, where, for simplicity of representation, it is assumed that $R^* = I_3$ and $\mathbf{p}^* = [0\ 0\ c]^T$, $c > 0$. The desired configuration is represented in black by the vector $\mathbf{p}^*$ and the coordinate frame $\{d\}$. The remaining configurations are represented in gray.

Lemma 3.1 reflects the topological obstacles, discussed in [5], [6], and [7], to achieving, by continuous state feedback, global stabilization of systems evolving on manifolds not diffeomorphic to the Euclidean Space. In fact, given a system evolving on a manifold $\mathcal{M}$, GAS of a single equilibrium point would imply the existence of a smooth positive definite function $V : \mathcal{M} \mapsto \mathbb{R}$ with negative definite derivative over all $\mathcal{M}$, that could be viewed as a Morse function with a single critical point, and, to admit such a function, $\mathcal{M}$ would have to be diffeomorphic to the Euclidean Space [7]. In view of these obstacles, a relaxation in the concept of GAS from global to almost global needs to be considered. It allows for the existence of a zero measure set of initial conditions that do not tend to the specified equilibrium point. In practical terms, this relaxation is fairly innocuous, since disturbances or noise will prevent trajectories from remaining at these (unstable) equilibria.

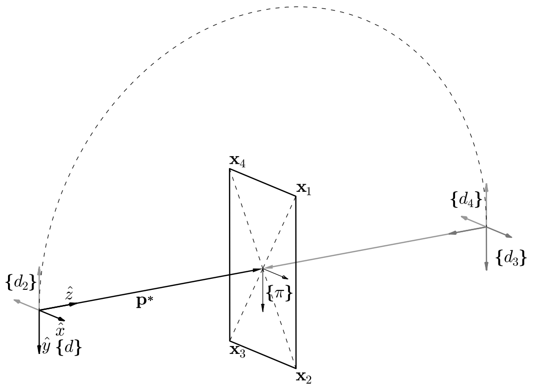
To formalize this concept of stability, which is adopted in [9], [10], and [3], we first recall the definition of region of attraction.

Definition 3.1 (Region of Attraction): Consider the autonomous system evolving on a smooth manifold $\mathcal{M}$

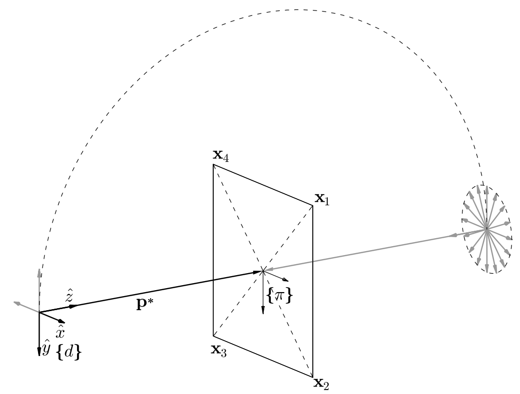
$$\dot{x} = f(x), \quad (18)$$

where $x \in \mathcal{M}$ and $f : \mathcal{M} \mapsto T\mathcal{M}$ is a locally Lipschitz manifold map, and suppose that $x = x^*$ is an asymptotically stable equilibrium point of the system. The region of attraction for x^* is defined as

$$\mathcal{R}_A = \{x_0 \in \mathcal{M} : \phi(t,x_0) \to x^* \text{ as } t \to \infty\}, \quad (19)$$



(a) rectangular configuration - four critical points



(b) square configuration - infinite number of critical points

Fig. 2. Critical points for two different landmark geometries.

where $\phi(t, x_0)$ denotes the solution of (18) with initial condition $x(0) = x_0$.

Definition 3.2 (Almost GAS): Consider the system (18). The equilibrium point $x = x^*$ is said to be almost globally asymptotically stable if it is stable and $\mathcal{M}\backslash\mathcal{R}_A$ is a set of zero measure.

Going back to the original kinematic model (5), we define the following continuous feedback law based on V

$$\mathbf{v} = k_v\mathbf{e} + k_\omega S(\mathbf{e} + \mathbf{p}^*)\mathbf{b}_\omega \tag{20a}$$

$$\boldsymbol{\omega} = k_\omega \mathbf{b}_\omega, \tag{20b}$$

where $\mathbf{b}_\omega = R^* S^{-1}(R_e X X^T - X X^T R_e^T)$. This control law will actually have the same equilibrium points as the simpler one considered before, but, in contrast, we will now be able to show that the "undesirable" equilibria are unstable and consequently that $(\mathbf{e}, R_e) = (0, I_3)$ is almost GAS. Additionally, we will see shortly that these control signals can be directly expressed in terms of the available measurements.

Theorem 3.2: For any k_v and k_ω positive, the closed-loop system resulting from the interconnection of (5) and (20) has an almost GAS equilibrium point at $(\mathbf{e}, R_e) = (0, I_3)$. The corresponding region of attraction is given by

$$\mathcal{R}_A = \{(\mathbf{e}, R_e) \in \mathrm{SE}(3) : \mathrm{tr}(I_3 - R_e) < 4\}\,. \tag{21}$$

Remark 3.1: To prove almost GAS, we need to determine the actual region of attraction and not just an estimate of it, as could be readily obtained by using Lyapunov's method. This would only provide us with a closed invariant set of the form $\Omega_c = \{x \in \mathcal{M} : V(x) \leq c\} \subset \mathcal{R}_A$. Instead, the proof of Theorem 3.2 relies on Zubov's theorem, which can be used to find the boundary of $\mathcal{R}_A$ (see [11] and [12]). For the sake of completeness, we restate the theorem, with only slight alterations to the version presented in [12].

Theorem 3.3 (Zubov's Theorem): Consider the system (18) and suppose that f is Lipschitz continuous on the region of attraction $\mathcal{R}_A$ of an asymptotically stable equilibrium point x^*. Then, an open set $\mathcal{G}$ containing x^* coincides with $\mathcal{R}_A$ if and only if there exist two continuous positive definite functions $W : \mathcal{G} \mapsto \mathbb{R}$ and $h : \mathcal{M} \mapsto \mathbb{R}$ such that

(i) $W(x^*) = 0$, $W(x) > 0$ for all $x \in \mathcal{G}\backslash\{x^*\}$,

(ii) $W(x) \to 1$ as $x \to \partial\mathcal{G}$ or, in the case of unbounded $\mathcal{G}$, as $d(x, x^*) \to \infty$, where $\partial\mathcal{G}$ is the boundary of $\mathcal{G}$ and $d(.,.)$ is a metric defined on $\mathcal{M}$,

(iii) $\dot{W}(x)$ is well defined for all $x \in \mathcal{G}$ and

$$\dot{W}(x) = -h(x)\,(1 - W(x))\,. \tag{22}$$

Proof: [Theorem 3.2.] We start by showing that the derivative of V is nonpositive. Substituting (20) in (11) and (12) yields

$$\dot{V} = -k_v n \mathbf{e}^T\mathbf{e} - k_\omega \mathbf{b}_\omega^T \mathbf{b}_\omega.$$

Then, we have $\dot{V} \leq 0$ for all $(\mathbf{e}, R_e) \in \mathrm{SE}(3)$ and $\dot{V} = 0$ for all $(\mathbf{e}, R_e) \in \mathcal{C}_V$, the set critical points of V determined in Lemma 3.1. By Lyapunov's stability theory, we can conclude local stability of $(0, I_3)$ and, by LaSalle's invariance principle, global convergence to $\mathcal{C}_V$. To prove almost global asymptotic stability of $(0, I_3)$, consider the continuously differentiable positive definite function

$$\bar{V}_2(R_e) = \mathrm{tr}(I_3 - R_e),$$

which corresponds to (9) with $X = I_3$. Using the angle-axis representation, $R_e = \mathrm{rot}(\theta, \mathbf{n})$, and with an obvious abuse of notation, $\bar{V}_2$ can be expressed as $\bar{V}_2(\theta) = 2(1 - \cos\theta)$. Using (14) with $P = 2I_3$ and (20b), the time derivative $\dot{\bar{V}}_2$ can be written as

$$\dot{\bar{V}}_2 = -2\sin\theta\,\mathbf{n}^T R^{*T}\boldsymbol{\omega} = -2k_\omega(\sin\theta)^2\,\mathbf{n}^T P\mathbf{n} \leq 0. \tag{23}$$

Defining the set $\mathcal{G} = \{R_e \in \mathrm{SO}(3) : \mathrm{tr}(I_3 - R_e) < 4\}$, it is straightforward to show that $W(R_e) = \frac{1}{4}\bar{V}_2(R_e)$ together with $h(R_e) = k_\omega V_2(R_e)$ satisfy the conditions of Theorem 3.3 and therefore $\mathcal{G} = \mathcal{R}_A$. By noting that $\mathcal{G}$ can also be written as $\mathcal{G} = \{\mathrm{rot}(\pi, \mathbf{n}) : \mathbf{n} \in \mathbb{S}^2\}$ and that, as stated in [5], the mapping from $R_e \in \mathrm{SO}(3)$ to the angle of rotation $\theta \in [0,\ \pi]$ defines a metric on $\mathrm{SO}(3)$, one concludes that $\mathcal{G}$ has zero measure. ■

Remark 3.2: When XX^T satisfies certain conditions, the function $V_2(R_e)$ defined in (9) corresponds to the modified trace function on $\mathrm{SO}(3)$ studied in [5] and [4]. In those works, to prove almost GAS of the desired equilibrium points, the authors rely on the fact that V_2 is a Morse function on $\mathrm{SO}(3)$, i.e. a function whose critical points are all nondegenerate and consequently isolated [5]. This corresponds to constraining P, or equivalently XX^T, to have all distinct eigenvalues. In our work, this restriction has been lifted, since the proof of almost GAS follows a different approach. As shown earlier, we can consider configurations (such as the square), which does not yield a Morse function for V_2, because the critical points can form the connected set $C_{V_2}^{(2)}$ given in (17).

A. Properties of the control law

The first property that we would like to highlight is that the control law (20) can be expressed solely in terms of the current and desired outputs $\mathbf{y}$ and $\mathbf{y}^*$, respectively. The following result establishes this.

Lemma 3.4: Under Assumption 2, the control law defined in (20) can be rewritten as

$$\mathbf{v} = k_v E\left(\mathbf{y} - \mathbf{y}^*\right) + S(E\mathbf{y})\boldsymbol{\omega} \tag{24a}$$

$$\boldsymbol{\omega} = k_\omega F(\mathbf{y}^*)\mathbf{y} - k_\omega n S(E\mathbf{y}^*)E\mathbf{y}, \tag{24b}$$

where $E = \frac{1}{n}[I_3 \cdots I_3] \in \mathbb{R}^{3\times 3n}$ and $F(\mathbf{y}^*) = [S(\mathbf{q}_1^*) \cdots S(\mathbf{q}_n^*)] \in \mathbb{R}^{3\times 3n}$.

Proof: According to Assumption 2 and (4), we have $\mathbf{p} = \frac{1}{n}\sum_j \mathbf{q}_j = E\mathbf{y}$, where $E = \frac{1}{n}[I_3 \cdots I_3] \in \mathbb{R}^{3\times 3n}$ and so (20a) can be rewritten as $\mathbf{v} = k_v\mathbf{e} + S(\mathbf{p})\boldsymbol{\omega} = k_v E\left(\mathbf{y} - \mathbf{y}^*\right) + S(E\mathbf{y})\boldsymbol{\omega}$. To obtain an alternative expression for (20b), note that

$$\begin{aligned} &-\mathbf{a}^T R^* S^{-1}(R_e XX^T - XX^T R_e^T) = \mathrm{tr}(S(R^{*T}\mathbf{a})R_e XX^T) \\ &= -\mathbf{a}^T R^* {\textstyle\sum_j} S(\mathbf{x}_j)R_e\mathbf{x}_j = -\mathbf{a}^T {\textstyle\sum_j} S(\mathbf{q}_j^* - \mathbf{p}^*)(\mathbf{q}_j - \mathbf{p}), \end{aligned}$$

for all $\mathbf{a} \in \mathbb{R}^3$. Then, (20b) can be rewritten as $\boldsymbol{\omega} = k_\omega \sum_j S\left(\mathbf{q}_j^*\right)\mathbf{q}_j - k_\omega n S(\mathbf{p}^*)\mathbf{p}$ and therefore as (24b). ■

The remaining properties relate to the dynamic behaviour of the closed-loop system, which can be rewritten as

$$\dot{\mathbf{e}} = -k_v\mathbf{e} \tag{25a}$$

$$\dot{R}_e = -k_\omega(R_e XX^T - XX^T R_e^T). \tag{25b}$$

We can immediately conclude that the proposed control law decouples the position and orientation errors systems and that the position subsystem (25a) has a global exponentially stable equilibrium point at $\mathbf{e} = 0$.

To analyze the stability and convergence properties of the orientation subsystem, it is convenient to consider the angle of rotation θ and axis of rotation $\mathbf{n}$ (recall that R_e can be written as $R_e = \mathrm{rot}(\theta, \mathbf{n})$). The expressions for $\dot{\theta}$ and $\dot{\mathbf{n}}$ are specified in the following Lemma, whose proof can be found in [8].

Lemma 3.5: Let $R_e \in \mathrm{SO}(3)$ be represented as a rotation of angle θ about the axis $\mathbf{n}$. Then, for $0 < |\theta| < \pi$, the time derivatives of θ and $\mathbf{n}$ can be written as

$$\dot{\theta} = -\mathbf{n}^T R^{*T}\boldsymbol{\omega} \tag{26}$$

$$\dot{\mathbf{n}} = \frac{1}{2}\left(\frac{\sin\theta}{1-\cos\theta}S(\mathbf{n}) + I_3\right)S(\mathbf{n})R^{*T}\boldsymbol{\omega}, \tag{27}$$

respectively.

Given the control law $\boldsymbol{\omega} = k_\omega R^* Q(\theta, \mathbf{n})P\mathbf{n}$, it is straightforward to show that, in closed-loop, (26) and (27) become

$$\dot{\theta} = -k_\omega \sin\theta \; \mathbf{n}^T P\mathbf{n} \tag{28}$$

$$\dot{\mathbf{n}} = k_\omega S(\mathbf{n})^2 P\mathbf{n}, \tag{29}$$

respectively.

Recalling that $P > 0$, we can immediately conclude from (28) that the proposed controller guarantees not only the convergence of θ to the origin, but also the decreasing monotonicity of $|\theta|$. Considering now (29), if all eigenvalues of P are equal, i.e. $P = \alpha I_3$ for some $\alpha > 0$, then $\dot{\mathbf{n}} = 0$ and so the convergence of R to R^* is achieved by rotating along a constant axis of rotation, which is determined by the initial condition of the system. On the other extreme case, where all the eigenvalues of P are distinct, we can divide the two-sphere $\mathbb{S}^2$ into the positive and negative half-spaces associated with the smallest eigenvalue of P and show that $\mathbf{n}$ converges to the corresponding eigenvector, with positive or negative sign depending on which of the half-spaces the system has started. The boundary between the two sets constitutes an invariant set of the system. The following result formalizes these considerations and also intermediate cases not yet discussed.

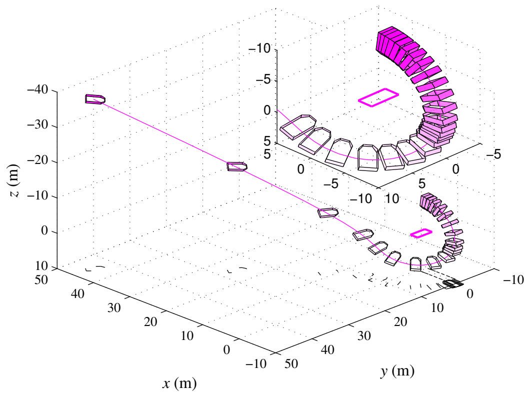
Lemma 3.6: ([8]) Let $P \in \mathbb{R}^{3\times 3}$ be the positive definite matrix, with eigenvalues $0 \le \lambda_1 \le \lambda_2 \le \lambda_3$. Then, (28) has an asymptotically stable equilibrium point at $\theta = 0$, with region of attraction $\{\theta : |\theta| < \pi\}$. Moreover, $|\theta|$ is monotonically decreasing. When the eigenvalues of P satisfy $\lambda_1 < \lambda_2 \le \lambda_3$, the asymptotically stable equilibrium points of (29) are given by the unitary eigenvectors $\mathbf{n}_1$ and $-\mathbf{n}_1$ associated with λ_1 and $\mathbf{n}(t) \to \mathrm{sign}(\mathbf{n}(0)^T\mathbf{n}_1)\mathbf{n}_1$ as $t \to \infty$, provided that $\mathbf{n}(0)^T\mathbf{n}_1 \ne 0$; when $\lambda_1 = \lambda_2 < \lambda_3$, the asymptotically stable equilibrium points form the set $\{\mathbf{n} : \mathbf{n} \in \mathrm{span}(\mathbf{n_1}, \mathbf{n}_2) \cap \mathbb{S}^2\}$ and the system converges to a point in this set provided that $\mathbf{n}(0) \ne \pm\mathbf{n}_3$.

This lemma turns out to be very useful, because it tells us how to select the axis of rotation to which $\mathbf{n}$ converges, by choosing the landmarks' placement.

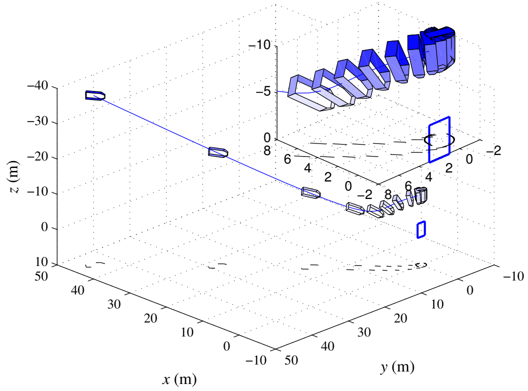
IV. SIMULATION RESULTS

In this section, we present simulation results that corroborate the stability characteristics of the system and illustrate the properties discussed in the previous section. We consider two different landmark configurations, corresponding to matrices $X_1 = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 2 & -2 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 \\ 2 & 2 & -2 & -2 \end{bmatrix}$. Figures 3(a) and (b) show the trajectories described by the system using control laws based on X_1 and X_2, respectively. Both were initialized at the same position and orientation and share the same target state $(\mathbf{p}^*, R^*) = ([0\ 0\ 10]^T, I_3)$.

We can see that, in both cases, the system starts by describing an almost straight-line trajectory in position, which reflects the quick convergence of $\mathbf{e}$ to a small neighborhood of the origin. From then on, the behaviour of the system is very much determined by the attitude controller, since the position evolves so as to keep $\mathbf{e}$ close to zero. At this point, the difference between trajectories becomes more pronounced. This behaviour is directly related to the placement of the measured points. As shown in Fig. 4, when X_1 is used, the axis of rotation converges to $[0\ -1\ 0]^T$ (dashed line) whereas when X_2 is used, it converges to $[0\ 0\ 1]^T$ (solid line). We recall that each of these vectors corresponds to the eigenvector associated with the smallest eigenvalue of



(a) Trajectory resulting from X_1.



(b) Trajectory resulting from X_2.

Fig. 3. System trajectories.

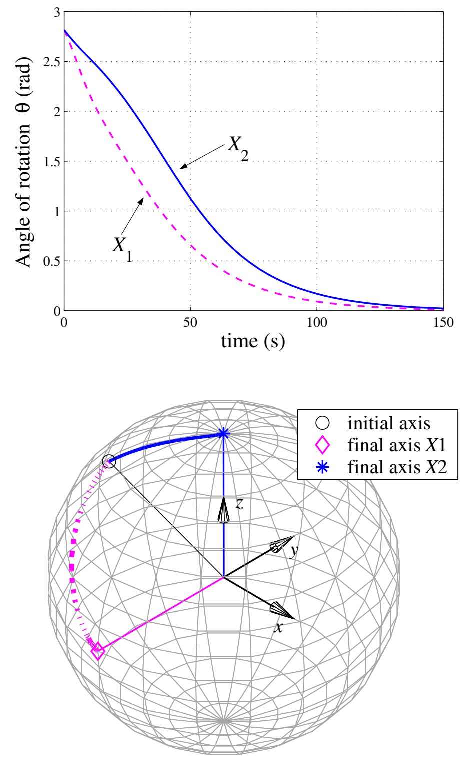


Fig. 4. Time evolution for the angle-axis pair $(\theta, \mathbf{n})$.

$P_1 = \mathrm{tr}(X_1 X_1^T)I_3 - X_1 X_1^T$ and $P_2 = \mathrm{tr}(X_2 X_2^T)I_3 - X_2 X_2^T$, respectively.

The obtained result suggests that a careful placement of the measured points with respect to the desired configuration can give rise to better-behaved trajectories. More specifically, if X is selected such that the axis of rotation converges to $\pm\mathbf{p}^*/\|\mathbf{p}^*\|$ (in the example, X_2 verifies this condition), the last stage of convergence will only involve a rotation about that axis, producing no translational motion.

V. CONCLUSIONS

The paper presented a solution to the problem of stabilization on $\mathrm{SE}(3)$. An output-feedback controller was defined, which guarantees almost global asymptotic stability of the desired equilibrium point. The output vector considered, which is formed by the body coordinates of a set of landmarks fixed in the environment, is relevant for a number of practical applications. The dependence of both the region of attraction and dynamic behaviour of closed-loop system on the geometry of the landmarks was specified. Future work will focus on extending these results to address the tracking problems and advance from the kinematic to a dynamic model.

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