A NONLINEAR FILTER FOR RANGE-ONLY ATTITUDE AND POSITION ESTIMATION

A. Alcocer [1] **P. Oliveira A. Pascoal**

Institute for Systems and Robotics and
Department of Electrical Engineering,
Instituto Superior Técnico,
Av. Rovisco Pais, 1096 Lisboa Codex, Portugal
e-mail: {alexblau,pjcro,antonio}@isr.ist.utl.pt

Abstract: A nonlinear filter structure is proposed to estimate the attitude and position of a vehicle using range measurements only. In the setup adopted, the vehicle is equipped with an array of beacons that determine their range to a set of landmarks with known locations. This scenario arises for instance in underwater acoustic navigation and GPS multiple antenna systems. We consider a simple discrete time kinematical model of the vehicle, the state of which can be identified with an element of the Special Euclidean group $\mathsf{SE}(3)$. The filter consists of a copy of the kinematic model of the moving body, plus a correction term which is biased towards the Maximum Likelihood (ML) estimate of its position and attitude based on current range measurements. In this sense, the proposed filter belongs to the class of Recursive Maximum Likelihood estimators and, as verified by simulation results, outperforms the static ML estimator even when the vehicle is describing unknown trajectories. In the framework adopted, the estimates evolve naturally on $\mathsf{SE}(3)$, thus eliminating the need for a normalization scheme that is recurrent in other formulations in flat Euclidean space. *Copyright ©2004 IFAC.*

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1. INTRODUCTION

Joint attitude and position estimation systems based on range measurements are becoming popular and have received the attention of the engineering community as an alternative to more complex, expensive, and sophisticated Inertial Navigation Systems. An advantage of such systems is that they are drift-less and insensible to magnetic disturbances. Examples of applications include multiple GPS receiver systems (Figure 1), indoor wireless network navigation systems, and acoustic systems to determine the attitude/position of a body underwater (Figure 2). See for ex-

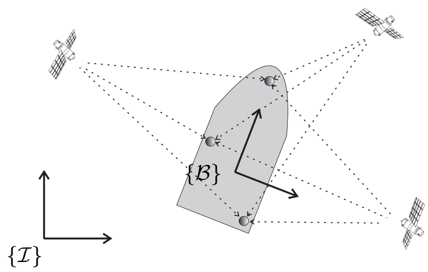

Fig. 1. GPS multi-antenna attitude/positioning system.

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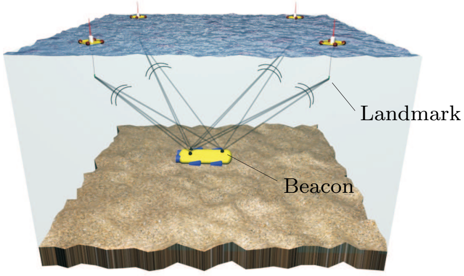

Fig. 2. Underwater acoustic attitude/positioning system with buoys as landmarks.

ample (Vickery, 1998), (Nadler *et al.*, 2000) and the references therein for some application examples. Range measurements are usually obtained by measuring the time it takes an electromagnetic or acoustic signal to travel between an emitter and a receiver given that the speed of propagation of the signals is assumed to be known.

In (Alcocer *et al.*, 2006) the authors derived a Maximum Likelihood (ML) estimator for the problem of attitude and position determination with range-only measurements. Supported on results on optimization on Riemannian manifolds (Edelman *et al.*, 1998), (Manton, 2002), generalized intrinsic gradient and Newton algorithms were derived to solve the problem at hand. The ML estimator uses observations from a single epoch. Past information, and the underlying vehicle equations of motion, are not taken into account in such scenario. This paper presents some preliminary results in the direction of designing a estimator that includes information about the vehicle kinematics.

In the case where measurements of the linear and angular velocities of a vehicle are available, given for instance by an inertial navigation system, it is of great interest to be able to fuse those measurements with range measurements. If no velocity measurements are available, it is also of interest to try to estimate the vehicle velocity from range measurements obtained at different instants of time. The paper proposes two nonlinear filters to solve both problems. The filters derived consider a copy of the vehicle kinematics plus a correction term "pointing" in the direction of the Maximum Likelihood estimate of the current measurements.

2. PROBLEM FORMULATION

2.1 Vehicle model

Let $\{\mathcal{B}\}$ and $\{\mathcal{I}\}$ denote a body-fixed reference frame and an inertial frame, respectively. Consider a simple vehicle discrete time kinematical model given by

$$\begin{cases} \mathbf{p}_{k+1} = \mathbf{p}_k + \mathbf{v}_k \\ \mathcal{R}_{k+1} = \mathcal{R}_k \exp(\mathcal{S}_k) \end{cases} \tag{1}$$

where $\mathbf{p} \in \mathbb{R}^3$ denotes the position of the origin of $\{\mathcal{B}\}$ expressed in $\{\mathcal{I}\}$, and $\mathbf{v} \in \mathbb{R}^3$ denotes the linear velocity of $\{\mathcal{B}\}$ with respect to $\{\mathcal{I}\}$ expressed in $\{\mathcal{I}\}$. The rotation matrix from $\{\mathcal{B}\}$ to $\{\mathcal{I}\}$, denoted $\mathcal{R}$, is an element of the Special Orthogonal group $\mathsf{SO}(3) = \{X \in \mathbb{R}^{3\times 3} : X^T X = I_3,\ \det(X) = +1\}$, where I_3 is the 3-dimensional identity matrix and $\det(\cdot)$ stands for the matrix determinant operator. The pair $(\mathbf{p}, \mathcal{R})$ is an element of the Special Euclidean group $\mathsf{SE}(3)$ defined in the sequel as the cartesian product $\mathbb{R}^3 \times \mathsf{SO}(3)$. The angular velocity $\omega = [\ \omega_1\ \omega_2\ \omega_3\ ]^T \in \mathbb{R}^3$ of $\{\mathcal{B}\}$ with respect to $\{\mathcal{I}\}$ expressed in $\{\mathcal{B}\}$ is represented in terms of the skew-symmetric matrix

$$\mathcal{S} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{2}$$

and in (1), $\exp(\cdot)$ denotes the matrix exponential. For simplicity, and without loss of generality, it is assumed that the filter sampling interval is equal to $h = 1s$. Note that the vehicle model in (1) can be interpreted as a step of unitary length along the geodesic emanating from $(\mathbf{p}_k, \mathcal{R}_k) \in \mathsf{SE}(3)$ in the direction given by the tangent vector $(\mathbf{v}_k, \mathcal{R}_k\mathcal{S}_k)$ (Murray *et al.*, 1994), (Park, 1995), (Žefran *et al.*, 1998).

2.2 Observations and the ML Estimate

Suppose that the vehicle has p beacons and assume that the location of the beacons with respect to $\{\mathcal{B}\}$ is known. The beacons could be GPS antennas, or acoustic emitters, arranged with a certain known geometry in the rigid body. Let us further consider that there are m fixed landmarks distributed in the ambient space with known positions. For example, GPS satellites or surface buoys equipped with hydrophones (see Figures 1 and 2). Let $\mathbf{b}_i \in \mathbb{R}^3, i \in \{1, \ldots, p\}$ denote the positions of the p beacons in the rigid body expressed in $\{\mathcal{B}\}$, and let $\mathbf{p}_j \in \mathbb{R}^3, j \in \{1, \ldots, m\}$ denote the positions of the m Earth fixed landmarks expressed in $\{\mathcal{I}\}$ (see Figure 3). Further let d_{ij} denote the distance between between beacon i and landmark j, defined by

$$d_{ij} = \|\mathcal{R}\mathbf{b}_i + \mathbf{p} - \mathbf{p}_j\| = \{(\mathbf{p} - \mathbf{p}_j)^T(\mathbf{p} - \mathbf{p}_j) + 2(\mathbf{p} - \mathbf{p}_j)^T\mathcal{R}\mathbf{b}_i + \mathbf{b}_i^T\mathbf{b}_i\}^{\frac{1}{2}} \tag{3}$$

with $i \in \{1, \ldots, p\}$, and $j \in \{1, \ldots, m\}$. Assume the vehicle has access to measurements y_{ij} defined by

$$y_{ij} = d_{ij} + w_{ij}. \tag{4}$$

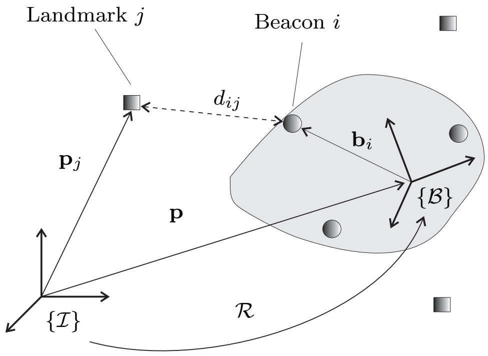

Fig. 3. Geometry of estimation problem.

consisting of the ranges d_{ij} corrupted by the additive Gaussian disturbances w_{ij}. [2]

It is convenient to stack all the ij elements in a more compact form by defining the vectors

$$\mathbf{d} \triangleq \left[d_{11}\ d_{21}\ \cdots\ d_{p1} \cdots\ d_{1m}\ \cdots\ d_{pm}\right]^T \in \mathbb{R}^{mp} \quad (5)$$

and $\mathbf{y}, \mathbf{w} \in \mathbb{R}^{mp}$ in a similar way. With this arrangement, the observations can then be written in a more compact form as

$$\mathbf{y} = \mathbf{d} + \mathbf{w}, \quad \mathbf{R} \triangleq \mathrm{E}\left\{\mathbf{w}\mathbf{w}^T\right\} \in \mathbb{R}^{mp \times mp} \quad (6)$$

where $\mathbf{R}$ is the covariance matrix of $\mathbf{w}$. Note that no assumption is made on the structure of $\mathbf{R}$, thus allowing for a number of interesting mission scenarios and configurations. For instance, the measurements of the distances between an Earth fixed landmark and all the body beacons can suffer from highly correlated disturbances. This might be due to the fact that the observations originate from signals that have travelled almost through the same propagation channel. In those cases the covariance matrix $\mathbf{R}$ is close to block diagonal.

Given a set of range measurements $\mathbf{y}$ corresponding to a fixed instant of time, the Maximum Likelihood (ML) estimate is defined as

$$\left(\widehat{\mathbf{p}}^*, \widehat{\mathcal{R}}^*\right) = \arg \min_{(\mathbf{p},\mathcal{R}) \in \mathsf{SE}(3)} \frac{1}{2}(\mathbf{y}-\mathbf{d})^T \mathbf{R}^{-1}(\mathbf{y}-\mathbf{d}), \quad (7)$$

where from (3) and (5), $\mathbf{d} = \mathbf{d}(\mathbf{p}, \mathcal{R})$. In (Alcocer *et al.*, 2006), generalized gradient and Newton iterative methods to minimize the cost function in (7) were derived. At each iteration, the algorithms performed a step along a geodesic with direction given by a gradient or Newton-like tangent vector $\Delta = (\Delta_p, \Delta_R)$. That is, Δ_p is some vector in $\mathbb{R}^3$, and $\Delta_R = \mathcal{R}\mathcal{S} \in \mathbb{R}^{3\times 3}$, where $\mathcal{R}$ is the current iterate rotation matrix estimate and $\mathcal{S}$ is some skew-symmetric matrix. Recall that the geodesic emanating from $(\mathbf{p}, \mathcal{R})$ in the direction $\Delta = (\Delta_p, \Delta_R)$ is given by $\gamma(t) = \left(\mathbf{p} + t\Delta_p, \mathcal{R}\exp(t\mathcal{R}^T\Delta_R)\right)$. In fact, this is true if we consider $\mathsf{SE}(3)$ equipped with its canonical Riemannian metric, see (Park, 1995) and (Žefran *et al.*, 1998) for further details. The ML estimator determines the most probable position and rotation given a set of measurements from a single instant of time and assuming no prior information. What happens if, as in our case, the vehicle is not static but moves along a trajectory? It is expected that by using some nonlinear filtering scheme considering past as well as present observations, an increase in the estimation performance can be obtained. Next, two nonlinear filters are proposed that, as verified in simulations, outperform the static ML estimator when the vehicle is describing an unknown trajectory.

[2] Note that the terms *beacon* and *landmark* are used to illustrate the problem but they do not impose any particular role (in terms of emitter/receiver). They should be simply viewed as *range measuring devices*

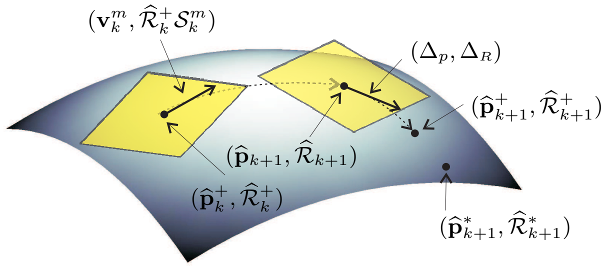

Fig. 4. Predict and Update filter steps

3. PROPOSED FILTER

Before addressing the pure Range-Only problem let us consider a frequent and conceptually simpler problem in which velocity measurements of the vehicle are assumed to be available.

3.1 Linear and Angular velocity measurements available

Suppose that we measure not only the ranges $\mathbf{y}$ but also the vehicle linear and angular velocities $\mathbf{v}^m$ and $\mathcal{S}^m$, respectively. Consider the following filter, which has the usual predict-update cycle structure found in Kalman Filtering:

Predict cycle:

$$\begin{cases} \widehat{\mathbf{p}}_{k+1} = \widehat{\mathbf{p}}_k^+ + \mathbf{v}_k^m \\ \widehat{\mathcal{R}}_{k+1} = \widehat{\mathcal{R}}_k^+ \exp(\mathcal{S}_k^m) \end{cases} \quad (8)$$

Update cycle:

$$\begin{cases} \widehat{\mathbf{p}}_k^+ = \widehat{\mathbf{p}}_k + k_p\Delta_p \\ \widehat{\mathcal{R}}_k^+ = \widehat{\mathcal{R}}_k \exp\left(k_R \widehat{\mathcal{R}}_k^T \Delta_R\right) \end{cases} \quad (9)$$

where $k_p, k_R \in \mathbb{R}$ are user defined gains. The correction term $\Delta = (\Delta_p, \Delta_R)$ used in the update step is a tangent vector that "points towards" the ML estimate of the current set of measurements (see Figure 4). Ideally, one could determine the ML estimate $(\widehat{\mathbf{p}}_k^*, \widehat{\mathcal{R}}_k^*)$ at each iteration and then define the correction term as $\Delta = (\widehat{\mathbf{p}}_k^* - \widehat{\mathbf{p}}_k, \widehat{\mathcal{R}}_k \mathrm{logm}(\widehat{\mathcal{R}}_k^T \widehat{\mathcal{R}}_k^*))$, where $\mathrm{logm}(\cdot)$ is the *log* operator on $\mathsf{SO}(3)$ (logm in MATLAB) which returns a skew-symmetric matrix whenever it is defined (Murray *et al.*, 1994). Note that by doing this and setting $k_p = k_R = 1$ a pure ML estimator would be obtained. If, on the other hand, we set $k_p = k_R = 0$ a pure prediction, or open loop integration of the velocity measurements would be obtained. Between those extreme values $0 \leq k_p, k_R \leq 1$ a compromise between pure ML and pure prediction would be obtained.

Instead of determining the ML estimate at each iteration, a simplified and computationally simpler version is proposed here, where the correction terms point towards the ML estimate in an iterative optimization based sense. That is, we consider Δ to be the intrinsic gradient or Newton descent direction of an ML iterative optimization scheme. The derivation of the intrinsic gradient and Newton descent directions are omitted due to space limitations and can be found in (Alcocer *et al.*, 2006).

3.2 Range-only measurements

Let us now consider the case in which only range measurements are available. In addition to the position and orientation pair $(\widehat{\mathbf{p}}, \widehat{\mathcal{R}})$, we will augment the state of the filter by including the linear and angular velocities $(\widehat{\mathbf{v}}, \widehat{\mathcal{S}})$. The proposed filter has the form:

Predict cycle:

$$\begin{cases} \widehat{\mathbf{p}}_{k+1} = \widehat{\mathbf{p}}_k^+ + \widehat{\mathbf{v}}_k \\ \widehat{\mathcal{R}}_{k+1} = \widehat{\mathcal{R}}_k^+ \exp(\widehat{\mathcal{S}}_k) \\ \widehat{\mathbf{v}}_{k+1} = \widehat{\mathbf{v}}_k^+ \\ \widehat{\mathcal{S}}_{k+1} = \widehat{\mathcal{S}}_k^+ \end{cases} \tag{10}$$

Update cycle:

$$\begin{cases} \widehat{\mathbf{p}}_k^+ = \widehat{\mathbf{p}}_k + k_p \Delta_p \\ \widehat{\mathcal{R}}_k^+ = \widehat{\mathcal{R}}_k \exp\left(k_R \widehat{\mathcal{R}}_k^T \Delta_{\mathcal{R}}\right) \\ \widehat{\mathbf{v}}_k^+ = \widehat{\mathbf{v}}_k + k_v \Delta_p \\ \widehat{\mathcal{S}}_k^+ = \widehat{\mathcal{S}}_k + k_S \widehat{\mathcal{R}}_k^T \Delta_R \end{cases} \tag{11}$$

where $k_p, k_R, k_v, k_S \in \mathbb{R}$ are user defined gains. It should not be surprising that the same correction term $\Delta = (\Delta_p, \Delta_R)$ is used both in determining $(\widehat{\mathbf{p}}^+, \widehat{\mathcal{R}}^+)$ and $(\widehat{\mathbf{v}}^+, \widehat{\mathcal{S}}^+)$. This is a common practice when designing Luenberger-like observers with output error injection. What it is not so common about the present approach is the way the correction terms are fed into the filter. Note that the correction terms are designed to lie on the tangent space to the current estimate, thus representing valid "movement directions". Note also that the filter is composed of a sequence of geodesic steps which ensure that the iterates will evolve naturally on $\mathsf{SE}(3)$. This eliminates the need of any normalization scheme. Moreover, since the filter uses no specific parametrization for $\mathsf{SE}(3)$, some problems such as singularities are avoided.

4. SIMULATION RESULTS

Simulations were performed to validate the proposed filters. Due to space limitations only the simulations with the Range-only filter in (10)-(11) are shown. In order to implement the filter, the Newton method was chosen to define the correction term (Δ_p, Δ_R). We considered an estimation setup where 8 landmarks were distributed at the corners of a 100m side cube. The vehicle was equipped with 3 beacons at positions given by the columns of $3I_3$m (where I_n is the $n \times n$ identity matrix). The measurement error covariance was set to $\mathbf{R} = \sigma^2 I_{mp}$ with $\sigma = 0.1$m. The filter (10)-(11) was implemented with user gains $k_p = 0.3$, $k_R = 0.3$, $k_v = 0.02$, and $k_S = 0.02$. The vehicle 3D trajectory during the simulations is shown in Figure 5. The actual and estimated vehicle position and attitude are shown in Figure 6. Exponential coordinates are used to represent the vehicle attitude. That is, the vector $\theta = [\theta_1 \ \theta_2 \ \theta_3]^T$ is used to represent the orthogonal matrix $\mathcal{R} = \exp\left\{ \begin{bmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{bmatrix} \right\}$. The actual and estimated linear and angular velocities are shown in Figure 7. The position and attitude estimation errors are shown in Figure 8. The attitude estimation errors are shown as the exponential coordinates of the error rotation $\widetilde{\mathcal{R}} = \widehat{\mathcal{R}}^T \mathcal{R}$. The linear and angular velocity estimation errors are shown in Figure 9.

5. PERFORMANCE EVALUATION

Given a set of noisy range measurements corresponding to a single epoch, there is a fundamental limitation on the size of the estimation errors that can be achieved. When the state space is Euclidean, the Cramér- Rao Bound (CRB) is the classic tool to determine such a limitation. When the state space is not Euclidean, as in our case, one should resort to generalizations of the CRB as the Intrinsic Variance Lower Bound (IVLB) (Xavier and Barroso, 2005). The IVLB uses the intrinsic (geodesic) Riemannian distance to quantify the estimation errors. More specifically, it sets a lower bound on the intrinsic variance of unbiased estimators. Let θ be the true parameter and $\widehat{\theta}$ an estimate of it. The intrinsic variance is defined as

$$\mathsf{var}\left\{\widehat{\theta}\right\} = \mathsf{E}\left\{d(\widehat{\theta}, \theta)^2\right\} \tag{12}$$

where $d(\cdot)$ is the intrinsic (geodesic) distance function. In the case of $\mathsf{SE}(3)$ with its canonical metric, we have (Park, 1995):

$$d_{\mathsf{SE}(3)}((\mathbf{p}_1, \mathcal{R}_1), (\mathbf{p}_2, \mathcal{R}_2)) = \sqrt{d_{\mathsf{SO}(3)}(\mathcal{R}_1, \mathcal{R}_2)^2 + \|\mathbf{p}_1 - \mathbf{p}_2\|^2}, \quad (13)$$

where

$$d_{\mathsf{SO}(3)}(\mathcal{R}_1, \mathcal{R}_2) = \sqrt{2} \arccos \left(\frac{\mathrm{tr}\left(\mathcal{R}_1^T \mathcal{R}_2\right) - 1}{2} \right) \quad (14)$$

In order to evaluate the performance of the proposed filter one could compare the estimation errors obtained at each iteration with the static ML estimator and with the IVLB. In order to do so, 100 Monte Carlo simulations were performed with the vehicle executing the same test trajectory. That is, at each point of the trajectory, 100 sets of independent noisy range measurements were generated with which a static ML estimator and the proposed filter were run. At each point the variance of the estimates produced with the ML estimator and the proposed filter were determined and compared to the corresponding IVLB, see Figure 10. Theoretically, if the sample size were large enough, the variance of the ML could never go beyond the IVLB. However, in practice this happens due to a small sample size. On the other hand, since the filter uses past as well as present measurements nothing prevents it from attaining lower variances than the IVLB. In Figure 10 it can be seen that after a transient, the proposed filter attains better performance than the ML and the IVLB.

6. CONCLUSIONS AND FUTURE WORK

The paper proposed two nonlinear filters for the problem of position and attitude determination with range-only measurements. The filters consist of a copy of the vehicle kinematics plus correction terms pointing towards the ML estimate of the current set of measurements. Due to the nature of the correction terms and the use of no particular parametrization, the iterates evolve naturally on $\mathsf{SE}(3)$ without requiring an extra normalization scheme and avoiding singularity problems. Simulation results show that by exploiting past as well as present measurements the proposed filter outperforms the static ML estimator when the vehicle is describing an unknown trajectory. Future work includes formal observability and stability analysis, and the derivation of design rules for the filter gains. Note that in order to fully evaluate the performance of the filters derived, one should use a generalization of the Posterior CRB (PCRB) instead of the IVLB only. This issue warrants further research.

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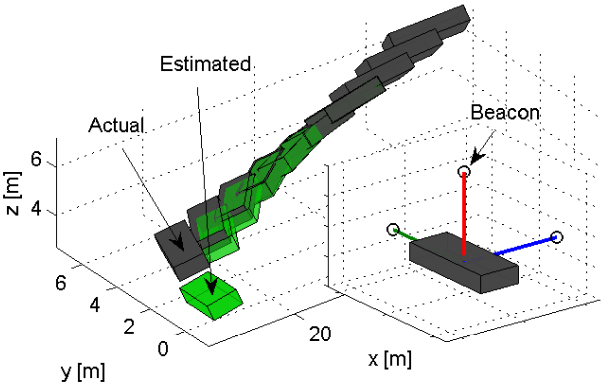

Fig. 5. Actual and estimated vehicle 3D trajectories. The vehicle together with body fixed beacons is shown on the right.

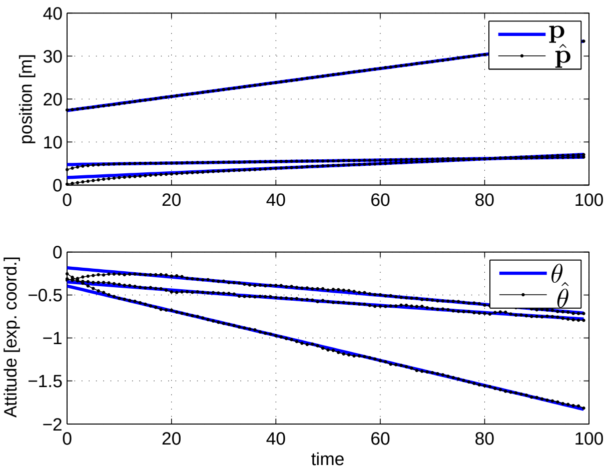

Fig. 6. Actual and estimated position / attitude. (Top) Entries of vectors $\mathbf{p}$ and $\widehat{\mathbf{p}}$. (Bot.) Entries of vectors θ and $\widehat{\theta}$, the exponential coordinates of $\mathcal{R}$ and $\widehat{\mathcal{R}}$, respectively.

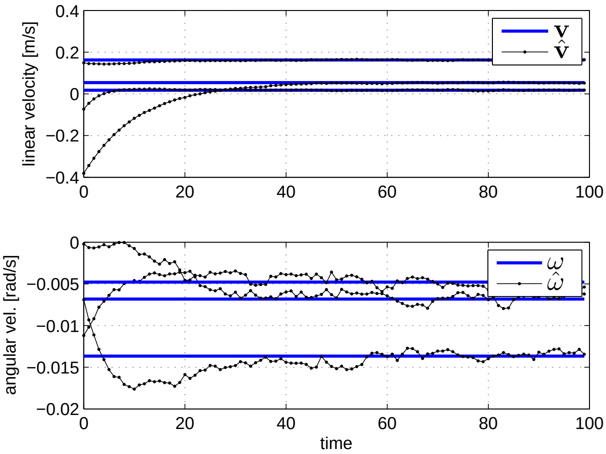

Fig. 7. Actual and estimated linear / angular velocities. (Top) Entries of vectors $\mathbf{v}$ and $\widehat{\mathbf{v}}$. (Bot.) Entries of vectors ω and $\widehat{\omega}$, obtained from the skew-symmetric matrices $\mathcal{S}$ and $\widehat{\mathcal{S}}$, respectively.

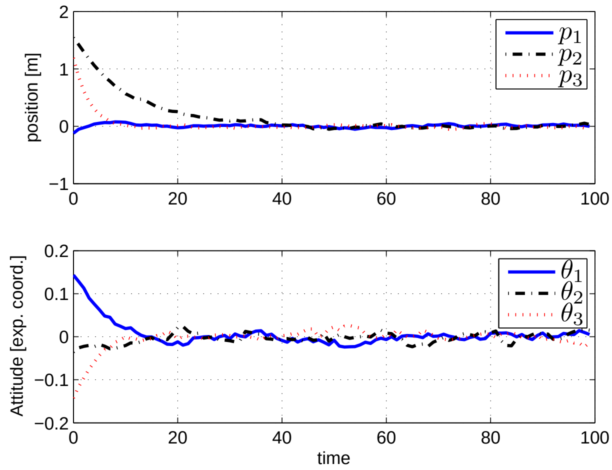

Fig. 8. Position / attitude estimation errors. (Top) Entries of vector $\widetilde{\mathbf{p}} = \mathbf{p} - \widehat{\mathbf{p}}$. (Bot.) Entries of vector $\widetilde{\theta}$, the exponential coordinates of $\widetilde{\mathcal{R}} = \widehat{\mathcal{R}}^T \mathcal{R}$.

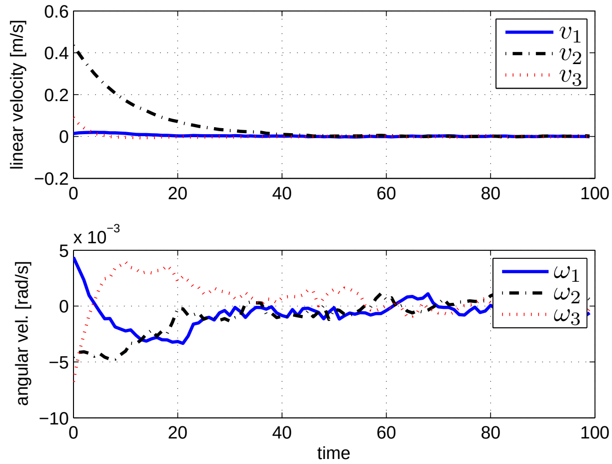

Fig. 9. Linear / angular velocity estimation errors. (Top) Entries of vector $\widetilde{\mathbf{v}} = \mathbf{v} - \widehat{\mathbf{v}}$. (Bot.) Entries of vector $\widetilde{\omega} = \omega - \widehat{\omega}$.

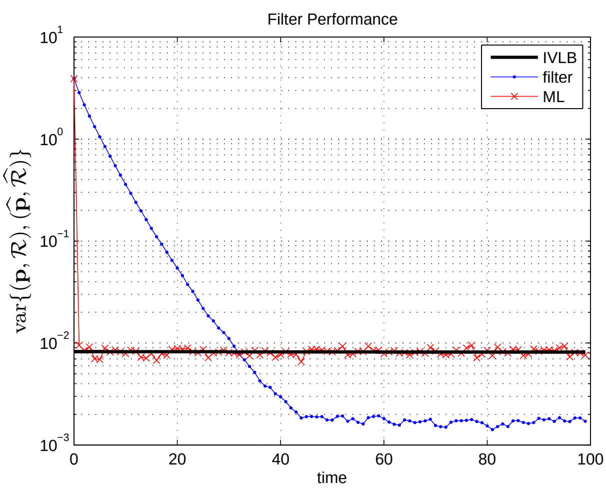

Fig. 10. Filter performance compared against the ML estimator and the IVLB.