CODEBOOK DESIGN FOR THE NON-COHERENT GLRT RECEIVER AND LOW SNR MIMO BLOCK FADING CHANNEL

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ABSTRACT

We address the problem of codebook design for the low signal-to-noise ratio (SNR) fast non-coherent MIMO block fading channel. The channel matrix is assumed deterministic (no stochastic model attached) and unknown at both the receiver and the transmitter. To handle the unknown deterministic space-time channel, a generalized likelihood ratio test (GLRT) receiver is implemented. The case of single transmit antenna is considered and it is shown that the problem of finding good codes corresponds geometrically to a packing problem in the complex projective space. We provide new constellations and demonstrate that they perform substantially better than state-of-art known solutions which assume equal prior probabilities for the transmitted codewords. Our results are also of interest for Bayesian receivers which decode constellations with non-uniform priors.

1. INTRODUCTION

In slowly fading scenarios, channel stability enables the receiver to be trained in order to acquire the channel state information (CSI) necessary for *coherent detection* of the transmitted codeword. Relying on the availability of CSI at the receiver, specific codebook design techniques have been introduced for coherent systems. In [1, 2], it has been shown that at high SNR the capacity of the multiple-antenna link increases linearly (when the rich scattering environment assumption holds) with the minimum number of transmitters and receivers. On the other hand, in fast fading scenarios, channel stability is lost, CSI is no more accessible, and the receiver must then operate in a *non-coherent* mode. It is known that the high SNR requirement implies low power efficiency which due to the power limitations in the mobile device cannot always be satisfied. This motivates the construction of communication schemes which can cope with the low SNR regime. See [3] for a more thorough discussion of this topic.

Previous work. In the literature, the problem of codebook design for noncoherent receivers facing low signal-to-noise (SNR) channels has been considered from two distinct points of view: the information-theoretic and the symbol error probability viewpoints. In either viewpoint, a statistical description of the channel is generally postulated. The particular case of an independent and identically distributed (iid) Rayleigh channel under an average power constraint has been analyzed from an information theoretic viewpoint in [4, 5]. The results in [4, 5] show that the capacity achieving input distribution becomes peaky at sufficiently low SNR. Jafar in [6] extended the characterization of the capacity achieving distribution for the correlated Rayleigh channel fading model. It is also known that, [4, 7], at low SNR, the mutual information is maximized by using only one transmit antenna.

The symbol error probability point of view for the analysis of low SNR non-coherent iid Rayleigh channel is more recent, although, Hochwald, et al. [8] had reported that in the low SNR and Rayleigh fading channel it seems one should employ only one transmit antenna. Borran et. al. [3], under the assumption of equally probable codewords, presented a technique that uses Kullback-Liebler (KL) divergence between the probability density functions induced at the receiver by distinct transmitted codewords as a design criterion for codebook design. In low SNR condition, their constellation points occupy multiple level (signal points lie in concentric spheres) with a point usually in the origin. The codes thereby constructed were shown to perform better than some existing non-coherent codebook constructions in low SNR, namely [8]. Recently, Srinivasan, et. al. [9], considered the case of single transmit antenna in the low SNR regime. Using the information theoretic results over the low SNR non-coherent iid Rayleigh fading channel under an average power constraint (c.f. [4, 5]), they allow for codewords with unequal priors in a code and optimize over prior probabilities to achieve better performance. This results in constellations that assume a point in the origin with probability $\frac{1}{2}$, with the probabilities of the points lying in the sphere being equal. By doing this, notable gain is reported as compared to codes designed with equal priors proposed by Borran. In [10], the correlated Rayleigh fading model was studied and it was shown that at any SNR, any single antenna performs better when used with suitable precoding in a MIMO correlated Rayleigh fading than in a single-input multiple output SIMO channel. Consequently, code designs that exploit the correlations in the transmit antennas in the MIMO case to provide gains over the corresponding SIMO case in the low SNR regime were presented.

Contribution. Contrary to other approaches for the low SNR regime, the channel matrix is assumed deterministic. We focus on $M = 1$ transmit antenna case. To handle the unknown space-time channel, a GLRT receiver is implemented. A low SNR analysis of the pairwise error probability (PEP) is introduced. We show that the problem of finding good codes corresponds to a packing problem in the complex projective space. New packings are designed and we demonstrate that our constellations perform substantially better than state-of-art known solutions which assume equal prior probabilities for the transmitted codewords. We also show that our codes can be incorporated in communication schemes with unequal priors.

Paper organization. In section 2, we describe the data model and our non-coherent receiver We introduce a low SNR analysis of PEP

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with a single transmit antenna in order to obtain a codebook design criterion. In Section 3, we present some codebook constructions and compare their performance with state-of-art solutions. Section 4 presents the main conclusions of our paper. Section 5 contains some mathematical details.

2. PROBLEM FORMULATION

Data model and assumptions. The communication system comprises $M = 1$ transmit and N receive antennas and we assume a block fading channel model with coherence interval T. In complex base band notation we have the model

$$\boldsymbol{Y} = \boldsymbol{x}\boldsymbol{h}^H + \boldsymbol{E},$$

where $\boldsymbol{x}$ is the $T \times 1$ vector of transmitted symbols (the vector $\boldsymbol{x}$ is called hereafter a codeword), $\boldsymbol{Y}$ is the $T \times N$ matrix of received symbols, $\boldsymbol{h}$ is the $N \times 1$ vector of channel coefficients, and $\boldsymbol{E}$ is the $T \times N$ matrix of zero-mean additive Gaussian observation noise. The symbol ^{H} denotes complex conjugate transpose. In $\boldsymbol{Y}$, time indexes the rows and space indexes the columns. We work under the following assumptions:

1. The channel vector $\boldsymbol{h}$ is not known at the receiver neither at the transmitter, and no stochastic model is assumed for it;
2. The codeword $\boldsymbol{x}$ is chosen from a finite codebook $\mathcal{X} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_K\}$ known to the receiver, where K is the size of the codebook. We impose the power constraint $||\boldsymbol{x}_k|| = \sqrt{\boldsymbol{x}_k^H \boldsymbol{x}_k} = 1$ for each codeword;
3. The observation noise is spatio-temporally white. In words, the noise covariance matrix is $\boldsymbol{\Upsilon} = \mathsf{E}[\text{vec}(\boldsymbol{E})\ \text{vec}(\boldsymbol{E})^H] = \boldsymbol{I}_{NT}$ (vec($\boldsymbol{E}$) stacks all columns of the matrix $\boldsymbol{E}$ on the top of each other, from left to right, and $\boldsymbol{I}_{NT}$ denotes the identity matrix of dimension $NT \times NT$). The matrix $\boldsymbol{\Upsilon}$ is known at the transmitter and at the receiver. Note that we have normalized the noise power. This entails no loss of generality.

GLRT receiver. Under the above assumptions, the conditional probability density function of the received vector $\boldsymbol{y} = \text{vec}(\boldsymbol{Y})$, given the transmitted vector $\boldsymbol{x}$, is given by

$$p(\boldsymbol{y}|\boldsymbol{x}, \boldsymbol{g}) = \frac{\exp\{-||\boldsymbol{y} - (\boldsymbol{I}_N \otimes \boldsymbol{x})\boldsymbol{g}||^2\}}{\pi^{TN}},$$

where $\boldsymbol{g} = \text{vec}\left(\boldsymbol{h}^H\right)$ is the unknown realization of the channel and $\otimes$ denotes Kronecker product.

Since no stochastic model is assumed for the channel propagation matrix, the receiver faces a multiple hypothesis testing problem where the channel $\boldsymbol{h}$ is a deterministic nuisance parameter. We assume a GLRT receiver which decides the index k of the codeword as

$$\widehat{k} = \underset{k = 1, 2, \ldots, K}{\text{argmax}} \quad p(\boldsymbol{y}|\boldsymbol{x}_k, \widehat{\boldsymbol{g}}_k)$$

where $\widehat{\boldsymbol{g}}_k = \left(\boldsymbol{I}_N \otimes \boldsymbol{x}_k^H\right)\boldsymbol{y}$. In words, the GLRT [11] consists in a bank of K parallel processors where the k-th processor computes the likelihood of the observation assuming the presence of the k-th codeword with the channel replaced by its maximum likelihood (ML) estimate.

Low SNR analysis. For the special case of unitary codebooks ($M > 1$) and spatio-temporal white Gaussian noise and iid Rayleigh fading, the exact expression and the Chernoff upper bound for the PEP have been derived in [12]. However, the calculus of these expressions for general non-coherent systems seems to be untractable. Instead, in this paper we resort to the PEP in low SNR regime.

Let $P_{\boldsymbol{x}_i \to \boldsymbol{x}_j}$ be the probability of the GLRT receiver deciding $\boldsymbol{x}_j$ when $\boldsymbol{x}_i$ is sent. It can be shown (details omitted) that

$$P_{\boldsymbol{x}_i \to \boldsymbol{x}_j} = P\left(X + Z > ||\boldsymbol{g}||^2 \sin^2 \alpha_{ij}\right) \tag{1}$$

where

$$X = \boldsymbol{e}^H \left(\boldsymbol{I}_N \otimes (\boldsymbol{\Pi}_j - \boldsymbol{\Pi}_i)\right) \boldsymbol{e}, \tag{2}$$

$Z = -2\,\Re\left(\boldsymbol{e}^H \boldsymbol{P}_{ij}\boldsymbol{g}\right)$, $\boldsymbol{P}_{ij} = \boldsymbol{I}_N \otimes \boldsymbol{\Pi}_j^{\perp}\boldsymbol{x}_i$ with $\boldsymbol{\Pi}_i = \boldsymbol{x}_i\boldsymbol{x}_i^H$ and $\boldsymbol{\Pi}_i^{\perp} = \boldsymbol{I}_T - \boldsymbol{\Pi}_i$. The operator $\Re(z)$ denotes the real part of the complex number z, and $\boldsymbol{e} = \text{vec}(\boldsymbol{E})$. The angle α_{ij} is the acute angle between the codewords $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$. Unfortunately, it seems that the PEP expression in (1) cannot be simplified, but we can analyze it at low SNR. At sufficiently low SNR, the noise quadratic term of $\boldsymbol{e}$ is the dominant one. Hence, we make the (admittedly crude) approximation

$$P_{\boldsymbol{x}_i \to \boldsymbol{x}_j} = P\left(X > ||\boldsymbol{g}||^2 \sin^2 \alpha_{ij}\right). \tag{3}$$

In the Appendix, section 5, we show that for $T \geq 2$

$$P\left(X > ||\boldsymbol{g}||^2 \sin^2 \alpha_{ij}\right) = P\left(\sum_{n=1}^{N}\left(|a_i|^2 - |b_i|^2\right) > ||\boldsymbol{g}||^2 \sin \alpha_{ij}\right), \tag{4}$$

where a_n, b_n are iid circular complex Gaussian random variables with zero mean and unit variance, $a_n, b_n \overset{iid}{\sim} \mathcal{CN}(0, 1)$ for $n = 1, \ldots, N$. Combining the expressions (3)-(4) we have the expression for the PEP at sufficiently low SNR (we assumed $\sin \alpha_{ij} \neq 0$)

$$P_{\boldsymbol{x}_i \to \boldsymbol{x}_j} = P\left(\sum_{i=1}^{N}\left(|a_i|^2 - |b_i|^2\right) > ||\boldsymbol{g}||^2 \sin \alpha_{ij}\right). \tag{5}$$

In our work [15] we derive the expression for the PEP in the high SNR regime. For $M = 1$, it is given by

$$P_{\boldsymbol{x}_i \to \boldsymbol{x}_j} = \mathcal{Q}\left(\frac{1}{\sqrt{2}}||\boldsymbol{g}|| \sin \alpha_{ij}\right) \tag{6}$$

where $\mathcal{Q}(x) = \int_x^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt$. Equations (5)-(6) show that the probability of misdetecting $\boldsymbol{x}_i$ for $\boldsymbol{x}_j$ depends on the channel $\boldsymbol{g} = \text{vec}\left(\boldsymbol{h}^H\right)$, but more important, on the relative geometry of the codewords $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$. Since $P_{\boldsymbol{x}_i \to \boldsymbol{x}_j} = P_{\boldsymbol{x}_j \to \boldsymbol{x}_i}$ (a feature of the scenario $M = 1$), the PEPs are symmetric which gives rise to a intuitive distance measure. Hence, by analyzing the PEP in both extreme cases (low and high SNR) it is clear that one wishes to make the codewords $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ as separate as possible, i.e., the problem of finding good codes corresponds to the very well known packing problem in the complex projective space [14].

3. RESULTS

Considering the results of the previous section, a codebook construction translates naturally into a packing problem in the complex projective space. Denoting a codebook by $\mathcal{X} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_K\}$ we are led to the following optimization problem

$$\mathcal{X}^* = \underset{\mathcal{X} \in \mathcal{M}}{\arg\max} \quad f(\mathcal{X}) \tag{7}$$

where $f\,:\,\mathcal{M}\rightarrow\mathbb{R}$, $\mathcal{X}=\{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_K\}\mapsto f(\mathcal{X})$ and

$$f(\mathcal{X})=\min\{f_{ij}(\mathcal{X}):1\leq i\neq j\leq K\}$$

with $f_{ij}(\mathcal{X}),=\boldsymbol{x}_i^H\boldsymbol{\Pi}_j^{\perp}\boldsymbol{x}_i$. The constraint space $\mathcal{M}=\{(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_K)$: $||\boldsymbol{x}_k||=1$ for all $k\}$ can be viewed as a multi-dimensional torus, i.e, the Cartesian product of K unit-spheres: $\mathcal{M}=\mathbb{S}^{2T-1}\times\cdots\times\mathbb{S}^{2T-1}$ (K times) and each codeword $\boldsymbol{x}_k$ belongs to $\mathbb{S}^{2T-1}$. Note that $f_{ij}(\mathcal{X})=\sin^2\alpha_{ij}$, hence, our goal is to make the codewords of the codebook as separate as possible. From (7), we see that the design of the codebook consists in a high-dimensional nonlinear nonsmooth optimization problem. To solve (7) we employ the algorithm presented in [15, 16]. Due to space constraints we just give a brief overview of the method presented therein. It contains two main steps. Step 1 starts by solving a convex SDP (Semi Definite Programming) relaxation to obtain a rough estimate of the optimal codebook. Step 2 refines it through a geodesic descent optimization algorithm which efficiently exploits the Riemannian geometry of the constraint space $\mathcal{M}$. Please refer to [15, 16] for more details.

We are not aware of any work concerning the low SNR noncoherent MIMO scenario employing a GLRT receiver. Hence, we shall compare the performance of our codes and our GLRT receiver with the codes assuming a Rayleigh fading channel with equally probable codewords [3] and ML receiver. We also show that our codes are of great interest for the constellations with unequal priors [9].

Constellations with equal priors. In all simulations we assume a Rayleigh fading model for the channel, i.e., $h_i\overset{iid}{\sim}\mathcal{CN}\left(0,\sigma^2\right)$. In figures 1- 2 we compared our codes and our GLRT receiver against the codes found in [3] with the ML receiver proposed therein. In figure 1, we considered the case where the coherence interval T=2, SNR=7 dB and a codebook with K=8 codewords. The solid and dashed curves represent our codes, and Borran codes respectively. As we can see, although the Borran's codes assume the knowledge of actual SNR $=\mathsf{E}\{||\boldsymbol{x}_k\boldsymbol{h}^H||^2\}/\mathsf{E}\{||\boldsymbol{E}||^2\}=7dB$, our codebook constructions can save up to 3 receive antennas at symbol error rate (SER) of $2\cdot10^{-3}$. The following $K\times T$ matrix (each row corresponds to a codeword) represents our codebook which was generated by the optimization algorithm in [15, 16]:

$$\begin{bmatrix}-0.6831+0.5082i & 0.4771+0.2179i\\ 0.1255-0.8888i & 0.1643+0.4090i\\ -0.6630+0.5108i & -0.1683+0.5208i\\ 0.2507+0.4084i & 0.2176+0.8503i\\ 0.5019-0.7276i & 0.4673-0.0211i\\ 0.4715-0.2047i & -0.8272+0.2269i\\ -0.5429+0.0964i & 0.0504-0.8327i\\ -0.0728+0.4296i & 0.8440+0.3128i\end{bmatrix}.$$

Figure 2 plots the results of a similar experiment for T=2, SNR=7 dB and K=16. It can be seen that for SER $=2\cdot10^{-2}$, our codes demonstrate a saving of 6 receive antennas when compared with Borran's codes. The following matrix represents our codebook, which we used in the figure 2,

$$\begin{bmatrix}-0.1424+0.7221i & 0.3490-0.5800i\\ 0.8352+0.4117i & 0.1901-0.3111i\\ 0.0091+0.1448i & -0.8493+0.5075i\\ 0.5890+0.7523i & -0.1535-0.2522i\\ -0.0178+0.7553i & 0.4480+0.4780i\\ 0.3061+0.4903i & -0.7628+0.2900i\\ 0.4328-0.0549i & 0.2059-0.8759i\\ -0.0694+0.5020i & -0.0147+0.8620i\\ 0.1163+0.3732i & 0.3510-0.8509i\\ 0.7910+0.3286i & -0.1021+0.5058i\\ 0.4420-0.8826i & 0.1290-0.0946i\\ 0.3624-0.7354i & 0.1788+0.5440i\\ -0.6304-0.3140i & -0.0999+0.7029i\\ 0.5160+0.0691i & -0.8253+0.2188i\\ -0.6139+0.0525i & -0.5840-0.5285i\\ 0.8232+0.0231i & 0.5597+0.0926i\end{bmatrix}.$$

Constellations with unequal priors. Now, we depart from our GLRT receiver and show that our codebook designs for $M=1$ are nevertheless of interest for schemes that allow for non-uniform priors. e.g.,the Bayesian receiver in [9]. In figure 3 we show the results of the simulations. We considered the case where the coherence interval T=2, SNR=0 dB and rate = 1 bps/Hz. The solid and dashed curves represent our codes, and Srinivasan's 5 point constellations with unequal priors [9] respectively. The dash-dotted curve represents our 4 point constellation with equal priors and is plotted only to confirm that if the receiver knows the channel statistics, then constellations with non-unifom priors are the best option. The gain of our 5 point constellations with unequal priors compared with Srinivasan's codes is due to the fact that we use optimal packings in complex projective space (in the outer sphere), whereas Srinivasan uses optimal packings in the real projective space (one can expect larger gains as K increases, where K represents the number of the codewords on the sphere). The improvement obtained can be explained by the optimality of our designed packings. Rankin bound is an upper bound on the packing radius of K subspaces in the Grassmanian space $G(M,\mathcal{C}^T)$. When $M=1$, the bound applies to packings in the projective space, and in this case it holds

$$\min\left\{\sin^2\alpha_{ij}:1\leq i\neq j\leq K\right\}\leq\frac{T-1}{T}\,\frac{K}{K-1}$$

where α_{ij} is the acute angle between codewords $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$. Please refer to [14] for more details. One can easily check that our designed codebook indeed meets the Rankin bound which is $\frac{2}{3}$ for $T=2$ and $K=4$. Our codebook is represented in the following matrix

$$\begin{bmatrix}0.4946-0.6268i & -0.2375+0.5533i\\ -0.8183-0.4446i & -0.3392+0.1328i\\ 0.4908-0.4101i & 0.7326+0.2329i\\ -0.0955-0.2776i & -0.8817+0.3693i\end{bmatrix}.$$

Constellations with equal priors and $M\geq1$. Finally, we present some results to study the impact of employing $M>1$ transit antennas in the low SNR regime. First, we compare our codebook constructions obtained by the method presented in [15, 16] for $M=1$ against Borran's codes with $M=2$. Next, we compare the scenarios $M=1$, $M=2$, $M=3$ using only our codes. We assume a Rayleigh fading model for the channel matrix, i.e., $h_{ij}\overset{iid}{\sim}\mathcal{CN}\left(0,\sigma^2\right)$. Figure 4 shows the result of the performance comparisons for 16 and 32-point constellations with $T=3$ and $T=4$, respectively, and SNR = 0 dB. The solid signed and the solid circled curve show the performance of our codes for K = 32, T = 4, M = 1, and K = 16, T = 3, M = 1, respectively. The dashed signed and the dashed circled curve represent the performances of the Borran's codes for K = 32, T = 4, M = 2 and K = 16, T = 3, M = 2, respectively. For 32-point constellation and at SER = $4 \cdot 10^{-2}$, we see that our codes can save 7 receive antennas. For 16-point constellation, we witness the gain of more than 10 receive antennas at SER = 10^{-1}. Figure 5 plots the result of the experiment for T=8, SNR=0 dB and K=256. It can be seen that for SER $= 2 \cdot 10^{-3}$, our codes for M=1 can spare 1 receive antennas when comparing with our codes constructed for M=2, and nearly 4 receive antennas compared with our codes constructed for M=3. We think that the results presented in the figures 4- 5 further strengthen the motivation of using a single transmit antenna codebooks in the low SNR regime when GLRT receiver is employed.

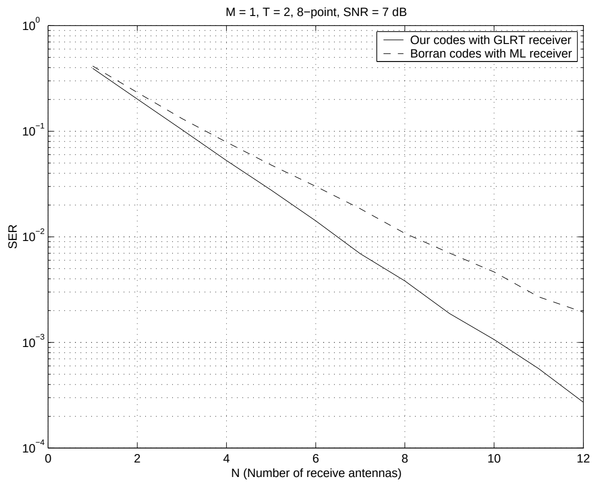

Fig. 1. M=1, T=2, K=8, SNR = 7 dB. Solid curve:our codes with our GLRT receiver. Dashed curve:Borran codes designed for SNR = 7dB with ML receiver [3].

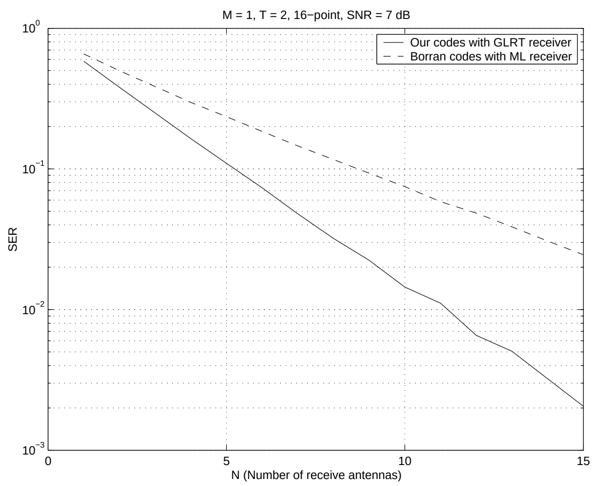

Fig. 2. M=1, T=2, K=16, SNR = 7 dB. Solid curve:our codes with our GLRT receiver. Dashed curve:Borran codes designed for SNR = 7dB with ML receiver [3].

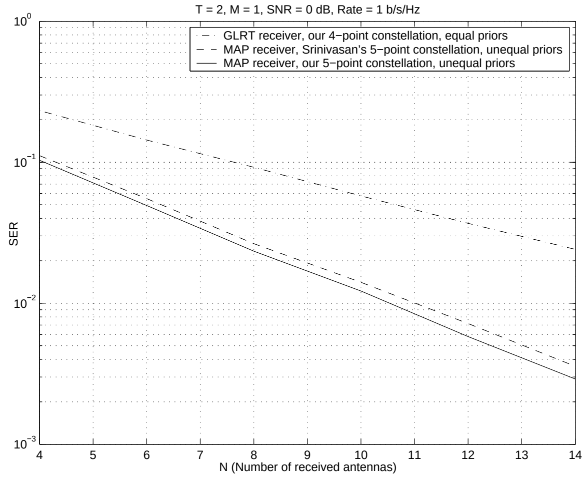

Fig. 3. T=2, M=1, SNR = 0 dB, Rate = 1 b/s/Hz. Solid curve-our 5 point constellation with unequal priors, dashed curve-Srinivasan's 5 point constellation with unequal priors [9], dash-dotted curve-our 4 point constellation with equal priors. Our and Srinivasan's 5 point constellations use *maximum a-posteriori* (MAP) receiver, our 4 point constellation uses GLRT receiver.

4. CONCLUSIONS

Contrary to other approaches for the low SNR regime, in this work, the channel matrix is assumed deterministic, i.e., no stochastic model is attached to it. To handle the unknown space-time channel, a generalized likelihood ratio test (GLRT) receiver is implemented. A low signal-to-noise (SNR) analysis of the pairwise error probability (PEP) and a single transmit antenna is introduced. We show that the problem of finding good codes corresponds to the very well known packing problem in the complex projective space. We provide some good packings and demonstrate that our constellations perform substantially better than state-of-art known solutions which assume equal prior probabilities, and are also of interest for the constellations with unequal priors.

Acknowledgement. The authors would like to thank Mr. Mohammad Jaber Borran for providing us the codes we use in the figure 4.

5. APPENDIX

In this appendix, we establish expression (4). We start by using the known fact from [13]: If $\boldsymbol{x}_i, \boldsymbol{x}_j \in \mathcal{C}^T$ such that $||\boldsymbol{x}_i|| = ||\boldsymbol{x}_j|| = 1$ ($T \geq 2$), then there exist unit-magnitude complex numbers u and v, and an $T \times T$ unitary matrix $\boldsymbol{Q}$ such that

$$\boldsymbol{Q}\boldsymbol{x}_i\, u = \begin{bmatrix} 1 \\ \mathbf{0}_{(T-1)\times 1} \end{bmatrix}, \;\; \boldsymbol{Q}\boldsymbol{x}_j\, v = \begin{bmatrix} \cos\alpha_{ij} \\ \sin\alpha_{ij} \\ \mathbf{0}_{(T-2)\times 1} \end{bmatrix}$$

where α_{ij} is the acute angle between the codewords $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$. Let $\boldsymbol{e}^T = \left[\boldsymbol{e}_1^T \boldsymbol{e}_2^T \ldots \boldsymbol{e}_N^T\right]$ with $\boldsymbol{e}_k \overset{iid}{\sim} \mathcal{CN}\left(\mathbf{0}, \boldsymbol{I}_T\right)$ for $k = 1, \ldots, N$. Now, it is not difficult to see that X defined in (2) satisfies

$$X \;\; \overset{d}{=} \;\; \sum_{k=1}^{N} \boldsymbol{c}_k^H \underbrace{\begin{bmatrix} -\sin^2\alpha_{ij} & \sin\alpha_{ij}\cos\alpha_{ij} \\ \sin\alpha_{ij}\cos\alpha_{ij} & \sin^2\alpha_{ij} \end{bmatrix}}_{\boldsymbol{Z}} \boldsymbol{c}_k,$$

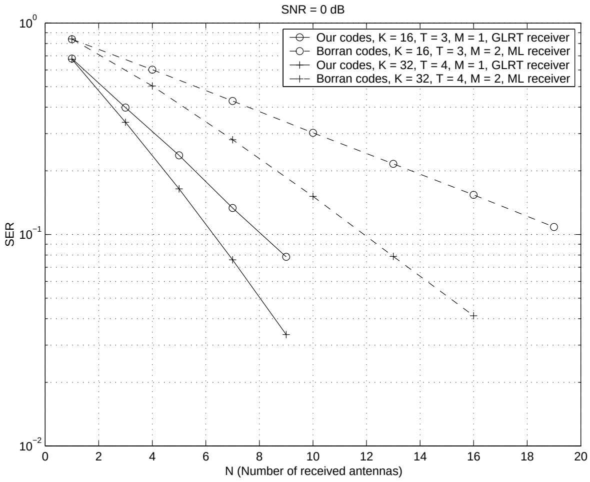

Fig. 4. Solid signed curve-our codes for K = 32, T = 4, M = 1, dashed signed curve-Borran codes for K = 32, T = 4, M = 2, solid circled curve-our codes for K = 16, T = 3, M = 1, dashed circled curve-Borran codes for K = 16, T = 3, M = 2.

where $\boldsymbol{c}_k \overset{iid}{\sim} \mathcal{CN}(\mathbf{0}, \boldsymbol{I}_2)$ for $k = 1, \ldots, N$, and $\overset{d}{=}$ means equal in distribution. Now, define $\boldsymbol{c}_k = \begin{bmatrix} a_k \\ b_k \end{bmatrix}$, with $a_k, b_k \overset{iid}{\sim} \mathcal{CN}(0, 1)$ and note that $\pm \sin \alpha_{ij}$ are the eigenvalues of the square symmetric matrix $\boldsymbol{Z}$. Hence, we shall have

$$X \overset{d}{=} \sum_{k=1}^{N} \left(|a_k|^2 - |b_k|^2\right) \sin \alpha_{ij}. \tag{8}$$

Combining (4) with (8) results in (5).

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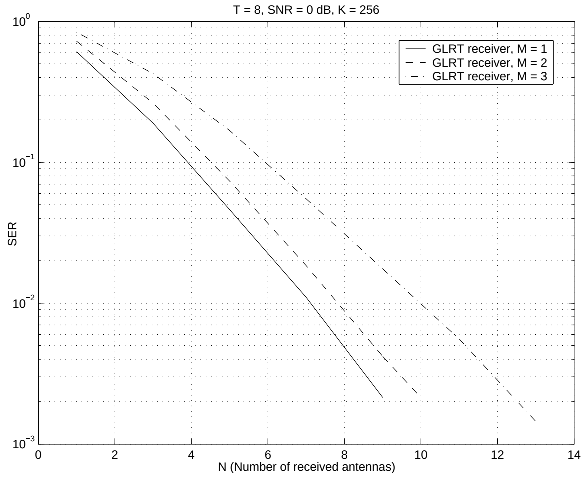

Fig. 5. T=8, K=256, SNR = 0 dB. Solid curve-our codes for M = 1, dashed curve-our codes for M = 2, dash-dotted curve-our codes for M = 3. All codes use GLRT receiver.

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